



UPSC PHYSICS OPTIONAL for CSE & IFoS

ABHI PHYSICS

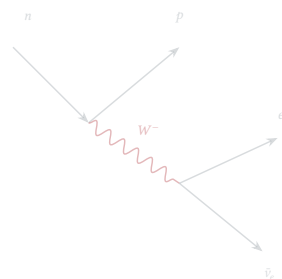
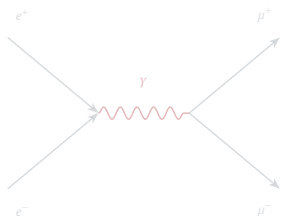
$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$E_n = -\frac{13.6}{n^2} \text{ eV}$$

PREVIOUS YEAR QUESTIONS



41 YEARS QUESTION BANK | 1986-2026

CSE + IFoS

SUBJECT

Waves and Optics

10 SOLVED QUESTIONS • CSE + IFoS

$$n \rightarrow p + e^- + \bar{\nu}_e$$

$$E^2 = p^2 c^2 + m^2 c^4$$

$$\partial_\mu j^\mu = 0$$

SUBJECT-WISE • TOPIC-WISE • SOURCE-TAGGED

For Civil Services and Indian Forest Service Examinations

Sample Preview extbar 10 Solved Questions

Question Index

No.	Question	Marks	Year	Exam	Page
1	A particle executes simple harmonic motion of amplitude A and angular frequency ω . A damping force proportional to velocity acts on the particle. Derive the expression for the displacement of the particle as a function of time and discuss the effect of damping on amplitude and frequency. Also, calculate the time at which the amplitude reduces to half its initial value, if the damping coefficient is b and mass is m .	10	2026	CSE	3
2	$m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = 0$; A harmonic oscillator is represented by the equation $m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = 0$; where $m = 0.25 \text{ kg}$, $\gamma = 0.07 \text{ kg s}^{-1}$ and $k = 85 \text{ Nm}^{-1}$. Determine (i) the period of oscillation, and (ii) the number of oscillations in which its amplitude will become half of its original value.	15	2025	CSE	4
3	A 15.0 cm violin string, fixed at both the ends, is vibrating with the largest wavelength. The speed of the wave in this string is 250 m/s , and the speed of sound in air is 350 m/s . What is the frequency and wavelength of the emitted sound wave?	8	2025	IFoS	6
4	A particle of mass m is suspended from a spring which has mass m and force constant k . Show that the oscillation frequency is given by $\omega = \frac{\sqrt{3}}{2} \omega_0$, where ω_0 is frequency of oscillation when spring is considered massless.	15	2025	IFoS	7
5	A weakly damped harmonic oscillator consisting of spring-mass system has the following parameters : Mass $m = 0.25 \text{ kg}$, Spring constant $k = 100 \text{ N m}^{-1}$, Damping coefficient $\gamma = 1 \text{ N s m}^{-1}$. A periodic force $F = 5 \cos \omega t$ (newton) is applied to the system. Determine (i) the amplitude of the oscillator at resonance and (ii) the Q -value of the oscillator.	15	2024	CSE	9
6	Consider a damped harmonic oscillator given by the equation $\ddot{x} + 2\mu\dot{x} + \omega_0^2 x = 0$. Obtain the displacements for (i) overdamped, (ii) underdamped and, (iii) critically damped motion.	15	2024	IFoS	11
7	Calculate the group and phase velocities for the wave packet corresponding to a relativistic particle.	8	2024	IFoS	14
8	Using Huygens' principle for a plane wave travelling from rarer medium 1 to a denser medium 2, show that $\frac{\sin i}{\sin r} = \frac{v_1}{v_2} = \frac{\mu_2}{\mu_1}$, where i and r are the angles of incidence and refraction, respectively. v_1, μ_1 and v_2, μ_2 are the velocities and refractive indices in media 1 and 2, respectively.	10	2023	CSE	16
9	Write equation for damped harmonic oscillations and obtain expression for logarithmic decrement. In a damped harmonic motion, the first amplitude is 10 cm , which reduces to 2 cm after 50 oscillations, each of period 4 seconds. Determine the logarithmic decrement. Also, calculate the number of oscillations in which the amplitude decreases to 25%.	20	2023	CSE	18
10	Find an expression for the average energy of a forced oscillator in the steady-state motion.	10	2023	IFoS	20

CSE • 2026 • 10 MARKS

Q1. A particle executes simple harmonic motion of amplitude A and angular frequency ω . A damping force proportional to velocity acts on the particle. Derive the expression for the displacement of the particle as a function of time and discuss the effect of damping on amplitude and frequency. Also, calculate the time at which the amplitude reduces to half its initial value, if the damping coefficient is b and mass is m .

INTRODUCTION:

Let $x(t)$ denote the displacement from equilibrium, with the positive x -direction chosen arbitrarily. The undamped angular frequency is ω , so the restoring force constant is

$$k = m\omega^2.$$

The velocity-dependent damping force is assumed to be linear and opposite to the instantaneous velocity:

$$F_d = -b\dot{x}, \quad b > 0.$$

Thus the motion is governed by Newton's second law applied in an inertial frame. Define the damping rate

$$\gamma = \frac{b}{2m}.$$

The initial amplitude A will mean the value of the exponentially decaying amplitude envelope at $t = 0$. The phase of the oscillator at that instant is denoted by ϕ .

SOLUTION:

1. The restoring and damping forces give

$$m\ddot{x} = -m\omega^2 x - b\dot{x},$$

or

$$\ddot{x} + \frac{b}{m}\dot{x} + \omega^2 x = 0.$$

Using $\gamma = b/(2m)$, this becomes

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0.$$

To solve it, take

$$x(t) = e^{rt}.$$

Substitution gives the characteristic equation

$$r^2 + 2\gamma r + \omega^2 = 0,$$

whose roots are

$$r = -\gamma \pm \sqrt{\gamma^2 - \omega^2}.$$

For the usual oscillatory, or underdamped, case

$$\gamma < \omega \quad \Longleftrightarrow \quad b < 2m\omega,$$

define the damped angular frequency

$$\omega_d = \sqrt{\omega^2 - \gamma^2} = \sqrt{\omega^2 - \frac{b^2}{4m^2}}.$$

The roots are then

$$r = -\gamma \pm i\omega_d.$$

Hence

$$x(t) = e^{-\gamma t} (C \cos \omega_d t + D \sin \omega_d t).$$

Writing $C = A \cos \phi$ and $D = -A \sin \phi$, the displacement is

$$x(t) = Ae^{-bt/(2m)} \cos(\omega_d t + \phi)$$

$$\omega_d = \sqrt{\omega^2 - \frac{b^2}{4m^2}}.$$

If the initial conditions are specified as $x(0) = A$ and $\dot{x}(0) = 0$, direct determination of the constants instead gives

$$x(t) = Ae^{-bt/(2m)} \left[\cos(\omega_d t) + \frac{b}{2m\omega_d} \sin(\omega_d t) \right].$$

This is equivalent to the amplitude-phase form with an appropriate constant phase and amplitude parameter.

In the zero-damping limit $b \rightarrow 0$,

$$\gamma \rightarrow 0, \quad \omega_d \rightarrow \omega,$$

and the solution reduces to ordinary simple harmonic motion, as required.

2. The time-dependent amplitude envelope is

$$A(t) = Ae^{-\gamma t} = Ae^{-bt/(2m)}.$$

Therefore damping causes the amplitude to decrease exponentially rather than remain constant. Since the mechanical energy of a harmonic oscillator is proportional to the square of its amplitude,

$$E(t) \propto A^2(t),$$

its slowly varying envelope decays as

$$E(t) = E(0)e^{-bt/m}.$$

This agrees with the fact that

$$\frac{dE}{dt} = -b\dot{x}^2 \leq 0,$$

so the damping force continuously removes mechanical energy.

Damping also lowers the oscillation frequency from ω to

$$\omega_d = \sqrt{\omega^2 - \frac{b^2}{4m^2}} < \omega.$$

For weak damping, $b \ll 2m\omega$, the binomial expansion gives

$$\omega_d = \omega \sqrt{1 - \frac{b^2}{4m^2\omega^2}} \approx \omega \left(1 - \frac{b^2}{8m^2\omega^2}\right).$$

Thus the reduction in frequency is second order in b for weak damping, whereas the amplitude-decay rate is first order in b .

At critical damping,

$$b = 2m\omega,$$

the two characteristic roots coincide and

$$x(t) = (C + Dt)e^{-\omega t}.$$

The particle returns to equilibrium without oscillating. For

$$b > 2m\omega,$$

the motion is overdamped:

$$x(t) = Ce^{(-\gamma + \sqrt{\gamma^2 - \omega^2})t} + De^{(-\gamma - \sqrt{\gamma^2 - \omega^2})t},$$

and again no oscillation frequency exists.

3. Let $t_{1/2}$ be the time at which the amplitude envelope becomes $A/2$. Then

$$Ae^{-bt_{1/2}/(2m)} = \frac{A}{2}.$$

Cancelling A and taking natural logarithms,

$$-\frac{bt_{1/2}}{2m} = \ln\left(\frac{1}{2}\right) = -\ln 2.$$

Therefore

$$t_{1/2} = \frac{2m}{b} \ln 2.$$

Since b/m has dimensions of inverse time, $2m/b$ has dimensions of time, as required.

CONCLUSION:

For underdamped motion, $b < 2m\omega$, the displacement is

$$x(t) = Ae^{-bt/(2m)} \cos \left[\sqrt{\omega^2 - \frac{b^2}{4m^2}} t + \phi \right].$$

The amplitude decreases exponentially according to

$$A(t) = Ae^{-bt/(2m)},$$

while the angular frequency is reduced from the undamped value ω to

$$\omega_d = \sqrt{\omega^2 - \frac{b^2}{4m^2}}.$$

At $b = 2m\omega$ the system is critically damped, and for $b > 2m\omega$ it is overdamped; in both cases the motion is non-oscillatory. The time required for the amplitude envelope to fall to one-half of its initial value is

$$t_{1/2} = \frac{2m \ln 2}{b},$$

measured in seconds when m is in kilograms and b is in kg s^{-1} .

CSE • 2025 • 15 MARKS

Q2. $m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = 0$; A harmonic oscillator is represented by the equation $m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = 0$; where $m = 0.25 \text{ kg}$, $\gamma = 0.07 \text{ kg s}^{-1}$ and $k = 85 \text{ Nm}^{-1}$. Determine (i) the period of oscillation, and (ii) the number of oscillations in which its amplitude will become half of its original value.

INTRODUCTION:

The equation describes a one-dimensional oscillator subject to a linear restoring force and viscous damping. The displacement x is measured from the equilibrium position along a fixed positive axis. The restoring force is $-kx$, while the damping force is

$$F_d = -\gamma \dot{x},$$

so it always acts opposite to the instantaneous velocity.

The motion is assumed to remain in the linear regime, with constant m , k , and γ , and with no external driving force. Define

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad \beta = \frac{\gamma}{2m},$$

where ω_0 is the undamped natural angular frequency and β is the amplitude-decay constant. For underdamped motion, $\beta < \omega_0$, and the damped angular frequency is

$$\omega_d = \sqrt{\omega_0^2 - \beta^2}.$$

The period is $T = 2\pi/\omega_d$. The number of oscillations required for the exponentially decaying amplitude to

become one-half is obtained from the amplitude envelope.

SOLUTION:

Dividing the equation of motion by m ,

$$\ddot{x} + \frac{\gamma}{m}\dot{x} + \frac{k}{m}x = 0.$$

Using

$$2\beta = \frac{\gamma}{m}, \quad \omega_0^2 = \frac{k}{m},$$

this becomes

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2x = 0.$$

For a trial solution $x = e^{rt}$, the characteristic equation is

$$r^2 + 2\beta r + \omega_0^2 = 0.$$

Hence

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}.$$

If $\beta < \omega_0$, the roots may be written as

$$r = -\beta \pm i\omega_d, \quad \omega_d = \sqrt{\omega_0^2 - \beta^2}.$$

The displacement is therefore

$$x(t) = A_0 e^{-\beta t} \cos(\omega_d t + \phi),$$

where A_0 and ϕ are fixed by the initial conditions. Thus the amplitude envelope is

$$A(t) = A_0 e^{-\beta t}.$$

The units are consistent:

$$\left[\frac{k}{m} \right] = \frac{\text{N m}^{-1}}{\text{kg}} = \frac{\text{kg s}^{-2}}{\text{kg}} = \text{s}^{-2},$$

and

$$\left[\frac{\gamma}{m} \right] = \frac{\text{kg s}^{-1}}{\text{kg}} = \text{s}^{-1}.$$

(i) The decay constant is

$$\beta = \frac{\gamma}{2m} = \frac{0.07 \text{ kg s}^{-1}}{2(0.25 \text{ kg})} = \frac{0.07}{0.50} \text{ s}^{-1} = 0.14 \text{ s}^{-1}.$$

The square of the undamped natural angular frequency is

$$\omega_0^2 = \frac{k}{m} = \frac{85 \text{ N m}^{-1}}{0.25 \text{ kg}} = 340 \text{ s}^{-2},$$

so that

$$\omega_0 = \sqrt{340} \text{ rad s}^{-1} = 18.439 \text{ rad s}^{-1}.$$

Since

$$\beta^2 = (0.14)^2 = 0.0196 \text{ s}^{-2} < 340 \text{ s}^{-2} = \omega_0^2,$$

the oscillator is strongly underdamped and executes oscillatory motion.

Its damped angular frequency is

$$\begin{aligned} \omega_d &= \sqrt{\omega_0^2 - \beta^2} \\ &= \sqrt{340 - 0.0196} \text{ rad s}^{-1} \\ &= \sqrt{339.9804} \text{ rad s}^{-1} \\ &= 18.4386 \text{ rad s}^{-1}. \end{aligned}$$

Therefore, the period of the damped oscillation is

$$\begin{aligned} T &= \frac{2\pi}{\omega_d} \\ &= \frac{2\pi}{18.4386 \text{ rad s}^{-1}} \\ &= 0.34076 \text{ s} \\ &\approx 0.341 \text{ s}. \end{aligned}$$

As a limiting check, when $\gamma \rightarrow 0$, one has $\beta \rightarrow 0$ and

$$T \rightarrow T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{18.439} \text{ s} \approx 0.34075 \text{ s}.$$

The computed damped period is very slightly greater than T_0 , as expected because damping reduces the angular frequency.

(ii) Let $t_{1/2}$ be the time at which the amplitude becomes one-half of its initial value. From

$$A(t) = A_0 e^{-\beta t},$$

the condition $A(t_{1/2}) = A_0/2$ gives

$$A_0 e^{-\beta t_{1/2}} = \frac{A_0}{2}.$$

After cancelling A_0 ,

$$e^{-\beta t_{1/2}} = \frac{1}{2}.$$

Taking natural logarithms,

$$-\beta t_{1/2} = -\ln 2,$$

and hence

$$\begin{aligned} t_{1/2} &= \frac{\ln 2}{\beta} \\ &= \frac{0.693147}{0.14 \text{ s}^{-1}} \\ &= 4.951 \text{ s}. \end{aligned}$$

If N denotes the number of oscillation periods completed in this time, then

$$N = \frac{t_{1/2}}{T}.$$

Therefore,

$$N = \frac{4.951 \text{ s}}{0.34076 \text{ s}} = 14.53.$$

Equivalently,

$$N = \frac{\omega_d \ln 2}{2\pi\beta} = \frac{(18.4386)(0.693147)}{2\pi(0.14)} = 14.53.$$

For a discrete-cycle plausibility check, after 14 complete oscillations,

$$\frac{A_{14}}{A_0} = e^{-\beta(14T)} = e^{-(0.14)(14)(0.34076)} \approx 0.513,$$

whereas after 15 complete oscillations,

$$\frac{A_{15}}{A_0} = e^{-\beta(15T)} = e^{-(0.14)(15)(0.34076)} \approx 0.489.$$

Thus the amplitude crosses $A_0/2$ between the fourteenth and fifteenth completed oscillations, consistently with $N = 14.53$.

CONCLUSION:

The oscillator is underdamped because

$$\beta = 0.14 \text{ s}^{-1} < \omega_0 = 18.439 \text{ rad s}^{-1}.$$

Its damped angular frequency and period are

$$\boxed{\omega_d = 18.4386 \text{ rad s}^{-1}}, \quad \boxed{T \approx 0.341 \text{ s}}.$$

The amplitude decays as

$$A(t) = A_0 e^{-0.14t},$$

because the damping force acts opposite to the velocity. It becomes half of its initial value after

$$\boxed{t_{1/2} \approx 4.95 \text{ s}},$$

corresponding to

$$\boxed{N \approx 14.53 \text{ oscillations}}.$$

Hence the amplitude becomes one-half between the fourteenth and fifteenth completed oscillations; if an integral number of completed oscillations is required, it is below one-half after approximately

$$\boxed{15 \text{ complete oscillations}}.$$

is 250 m/s, and the speed of sound in air is 350 m/s. What is the frequency and wavelength of the emitted sound wave?

INTRODUCTION:

The required quantities are the frequency f_{sound} and wavelength λ_{sound} of the sound emitted into air.

Let

$$L = 15.0 \text{ cm} = 0.150 \text{ m}$$

$$v_s = 250 \text{ m s}^{-1}$$

$$v_a = 350 \text{ m s}^{-1}$$

where v_s is the transverse-wave speed on the string and v_a is the speed of sound in air.

Choose the x -axis along the string, with $x = 0$ at the left fixed end and $x = L$ at the right fixed end. The transverse displacement is denoted by $y(x, t)$, with positive y upward. Since both ends are fixed,

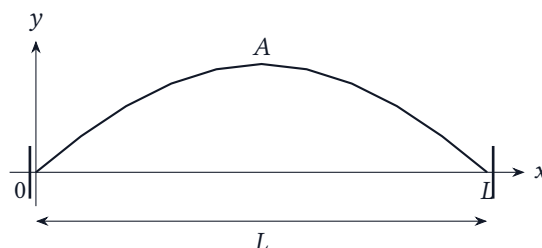
$$y(0, t) = y(L, t) = 0.$$

Thus, both ends must be nodes of every allowed standing-wave mode. The largest allowed wavelength is therefore the wavelength of the fundamental mode, for which the string contains one half-wavelength.

It is assumed that the string is uniform, the oscillations are small, the air is stationary and approximately nondispersive, and losses and end corrections are negligible. The string forces the adjacent air to oscillate at the same frequency; hence the sound frequency equals the vibration frequency of the string, although its wavelength changes because the wave speed changes on passing from the string to air.

SOLUTION:

For the fundamental standing-wave mode, the fixed ends are consecutive nodes and there is one antinode at $x = L/2$:



In general, if n half-wavelengths fit into a string of length L ,

$$L = n \frac{\lambda_n}{2},$$

so that

$$\lambda_n = \frac{2L}{n}, \quad n = 1, 2, 3, \dots$$

Q3. A 15.0 cm violin string, fixed at both the ends, is vibrating with the largest wavelength. The speed of the wave in this string

The wavelength is largest for the smallest permitted positive integer, $n = 1$. Hence the fundamental wavelength on the string is

$$\lambda_s = 2L.$$

Substituting the string length,

$$\lambda_s = 2(0.150 \text{ m}) = 0.300 \text{ m}.$$

1. The frequency of vibration of the string is obtained from $v = f\lambda$:

$$f_s = \frac{v_s}{\lambda_s} = \frac{250 \text{ m s}^{-1}}{0.300 \text{ m}} = 8.33 \times 10^2 \text{ s}^{-1}.$$

Therefore,

$$f_s = 8.33 \times 10^2 \text{ Hz}.$$

The air is driven periodically by the string. Since the number of oscillations produced per second cannot change at the string-air coupling,

$$f_{\text{sound}} = f_s = 8.33 \times 10^2 \text{ Hz}.$$

2. The wavelength of the sound in air is

$$\lambda_{\text{sound}} = \frac{v_a}{f_{\text{sound}}}.$$

On substitution,

$$\lambda_{\text{sound}} = \frac{350 \text{ m s}^{-1}}{8.33 \times 10^2 \text{ s}^{-1}} = 4.20 \times 10^{-1} \text{ m}.$$

Thus,

$$\lambda_{\text{sound}} = 0.420 \text{ m} = 42.0 \text{ cm}.$$

The dimensional checks are

$$[f] = \frac{[v]}{[\lambda]} = \frac{\text{m s}^{-1}}{\text{m}} = \text{s}^{-1} = \text{Hz}$$

$$[\lambda_{\text{sound}}] = \frac{[v_a]}{[f]} = \frac{\text{m s}^{-1}}{\text{s}^{-1}} = \text{m}.$$

The result is also physically plausible: the frequency remains unchanged, while

$$\frac{\lambda_{\text{sound}}}{\lambda_s} = \frac{v_a}{v_s} = \frac{350}{250} = 1.40.$$

Therefore, because sound travels 1.40 times faster in air than the transverse wave travels on the string, its wavelength in air is 1.40 times the standing-wave wavelength on the string:

$$1.40(0.300 \text{ m}) = 0.420 \text{ m}.$$

CONCLUSION:

For the largest-wavelength, fixed-end mode, the string vibrates in its fundamental mode with

$$\lambda_s = 2L = 0.300 \text{ m}.$$

The emitted sound has frequency

$$f_{\text{sound}} = 8.33 \times 10^2 \text{ Hz} \approx 833 \text{ Hz}$$

and wavelength in air

$$\lambda_{\text{sound}} = 0.420 \text{ m} = 42.0 \text{ cm}.$$

The sound propagates outward through the air from the vibrating string; no unique propagation direction is specified by the question. These results apply for a uniform string vibrating in the fundamental mode and for stationary, approximately nondispersive air with negligible end corrections and losses. The physical inference is that coupling to air preserves the frequency, while the higher wave speed in air produces a correspondingly larger wavelength.

IFoS • 2025 • 15 MARKS

Q4. A particle of mass m is suspended from a spring which has mass m and force constant k . Show that the oscillation frequency is given by $\omega = \frac{\sqrt{3}}{2}\omega_0$, where ω_0 is frequency of oscillation when spring is considered massless.

INTRODUCTION:

The required quantity is the angular frequency of small vertical oscillations when the inertia of the spring is included. Let the suspended particle have mass M , let the uniform spring have mass m_s , natural or equilibrium-coordinate length L , and force constant k . At the end we shall use the data

$$M = m_s = m.$$

Let y , measured vertically downward from the fixed end, label a point of the spring, so that $0 \leq y \leq L$. Let $x(t)$ be the small downward displacement of the lower end, and hence of the suspended particle, from the static equilibrium position. Measuring displacement from static equilibrium removes all terms linear in x arising from gravity; gravity changes the equilibrium extension but not the small-oscillation frequency.

For the standard effective-mass treatment of a uniform spring, the incremental longitudinal extension

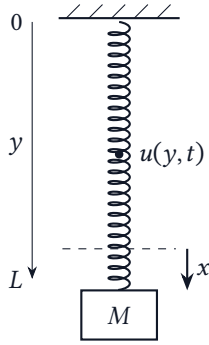
is assumed to be distributed uniformly. Since the upper end is fixed and the lower end moves through $x(t)$, the displacement field is

$$u(y, t) = \frac{y}{L}x(t), \quad u(0, t) = 0, \quad u(L, t) = x(t).$$

Thus different parts of the massive spring have different velocities and the entire spring mass cannot be added directly to M . The governing principle is conservation of mechanical energy, or equivalently Lagrange's equation, with both the particle's kinetic energy and the distributed kinetic energy of the spring included.

SOLUTION:

The geometry and sign convention are shown below. The coordinate y and the displacement x are both positive vertically downward.



Since the spring is uniform, its linear mass density is

$$\lambda = \frac{m_s}{L}.$$

An element of length dy therefore has mass

$$dm = \lambda dy = \frac{m_s}{L} dy.$$

From the assumed displacement field,

$$u(y, t) = \frac{y}{L}x(t),$$

the velocity of this element is

$$v(y, t) = \frac{\partial u}{\partial t} = \frac{y}{L}\dot{x}(t).$$

The element at the fixed end has zero velocity, while the element at the lower end has velocity $\dot{x}$, as required by the boundary conditions.

The kinetic energy of the spring is consequently

$$T_s = \frac{1}{2} \int_0^L v^2 dm.$$

Substituting $v = (y/L)\dot{x}$ and $dm = (m_s/L)dy$,

$$\begin{aligned} T_s &= \frac{1}{2} \int_0^L \left(\frac{y}{L} \dot{x} \right)^2 \frac{m_s}{L} dy \\ &= \frac{m_s \dot{x}^2}{2L^3} \int_0^L y^2 dy \\ &= \frac{m_s \dot{x}^2}{2L^3} \left[\frac{y^3}{3} \right]_0^L \\ &= \frac{m_s \dot{x}^2}{2L^3} \frac{L^3}{3} = \frac{1}{2} \left(\frac{m_s}{3} \right) \dot{x}^2. \end{aligned}$$

Thus, for this mode of motion, the spring contributes the effective inertial mass

$$m_{\text{eff, spring}} = \frac{m_s}{3}.$$

The kinetic energy of the suspended particle is

$$T_M = \frac{1}{2} M \dot{x}^2.$$

Hence the total kinetic energy is

$$T = T_M + T_s = \frac{1}{2} \left(M + \frac{m_s}{3} \right) \dot{x}^2.$$

The elastic potential energy can also be obtained directly from the distributed deformation. A spring segment of equilibrium length dy has force constant

$$k_y = k \frac{L}{dy},$$

because the force constant of a uniform spring is inversely proportional to its length. Its incremental extension is

$$du = \frac{\partial u}{\partial y} dy = \frac{x}{L} dy.$$

Therefore its elastic energy is

$$dV = \frac{1}{2} k_y (du)^2 = \frac{1}{2} \left(k \frac{L}{dy} \right) \left(\frac{x}{L} dy \right)^2 = \frac{1}{2} \frac{kx^2}{L} dy.$$

Integrating over the spring gives

$$V = \int_0^L dV = \frac{1}{2} \frac{kx^2}{L} \int_0^L dy = \frac{1}{2} kx^2.$$

The Lagrangian is therefore

$$\mathcal{L} = \frac{1}{2} \left(M + \frac{m_s}{3} \right) \dot{x}^2 - \frac{1}{2} kx^2.$$

Lagrange's equation,

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{\partial \mathcal{L}}{\partial x} = 0,$$

gives

$$\left(M + \frac{m_s}{3} \right) \ddot{x} + kx = 0.$$

Comparing this with the simple-harmonic equation

$$\ddot{x} + \omega^2 x = 0,$$

we obtain the general result

$$\omega^2 = \frac{k}{M + m_s/3}, \quad \omega = \sqrt{\frac{k}{M + m_s/3}}.$$

In the present problem,

$$M = m_s = m,$$

and hence

$$\begin{aligned} \omega &= \sqrt{\frac{k}{m + m/3}} \\ &= \sqrt{\frac{k}{4m/3}} \\ &= \sqrt{\frac{3k}{4m}} \\ &= \frac{\sqrt{3}}{2} \sqrt{\frac{k}{m}}. \end{aligned}$$

When the spring is considered massless, $m_s = 0$, so the angular frequency is

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

Consequently,

$$\omega = \frac{\sqrt{3}}{2} \omega_0.$$

The dimensional check is

$$\left[\frac{k}{m} \right] = \frac{\text{N m}^{-1}}{\text{kg}} = \frac{\text{kg s}^{-2}}{\text{kg}} = \text{s}^{-2},$$

so ω has dimension s^{-1} , conventionally expressed in rad s^{-1} .

As a limiting check, the general expression satisfies

$$\lim_{m_s \rightarrow 0} \sqrt{\frac{k}{M + m_s/3}} = \sqrt{\frac{k}{M}},$$

which is exactly the massless-spring result. Also,

$$\frac{\omega}{\omega_0} = \frac{\sqrt{3}}{2} \approx 0.866 < 1,$$

which is physically plausible: the moving spring adds inertia without increasing the force constant, and therefore lowers the frequency. During the oscillation the energy

$$E = \frac{1}{2} \left(M + \frac{m_s}{3} \right) \dot{x}^2 + \frac{1}{2} kx^2$$

is conserved.

CONCLUSION:

For small vertical oscillations of a uniform linear spring, with the upper end fixed and the incremental displacement varying as $u(y, t) = yx(t)/L$, the spring's distributed mass m_s contributes an effective oscillating mass $m_s/3$. Therefore

$$\omega = \sqrt{\frac{k}{M + m_s/3}}.$$

For a suspended particle and spring each of mass m ,

$$\omega = \sqrt{\frac{3k}{4m}} = \frac{\sqrt{3}}{2} \omega_0, \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

The motion is simple harmonic in the vertical direction about the static equilibrium position, and ω is measured in rad s^{-1} . The result is valid for small longitudinal oscillations of a uniform Hookean spring under the linear displacement-profile approximation. It shows that the inertia of the spring reduces the angular frequency to approximately 86.6% of its massless-spring value.

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Q5. A weakly damped harmonic oscillator consisting of spring-mass system has the following parameters : Mass $m = 0.25 \text{ kg}$, Spring constant $k = 100 \text{ N m}^{-1}$, Damping coefficient $\gamma = 1 \text{ N s m}^{-1}$. A periodic force $F = 5 \cos \omega t$ (newton) is applied to the system. Determine (i) the amplitude of the oscillator at resonance and (ii) the Q -value of the oscillator.

INTRODUCTION:

The required quantities are the maximum steady-state displacement amplitude under harmonic forcing and the quality factor of the oscillator. Let $x(t)$ denote displacement along the spring axis, with positive x chosen in the direction of the applied force at $t = 0$. The equation of motion is

$$m\ddot{x} + \gamma\dot{x} + kx = F_0 \cos \omega t, \quad F_0 = 5 \text{ N}.$$

Here the damping force is $-\gamma\dot{x}$, so that it always opposes the velocity.

Define the undamped natural angular frequency and the damping parameter by

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad \beta = \frac{\gamma}{2m}.$$

The weak-damping assumption requires

$$\beta \ll \omega_0 \iff \gamma \ll 2\sqrt{mk}.$$

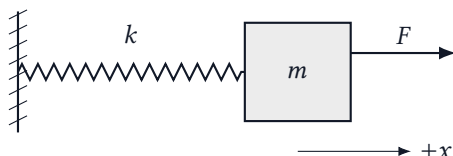
The steady-state displacement is written as

$$x(t) = A(\omega) \cos(\omega t - \delta),$$

where $A(\omega)$ is the non-negative amplitude and δ is the displacement lag relative to the applied force. Transient motion is omitted because it decays as $e^{-\beta t}$.

SOLUTION:

The physical arrangement and the adopted positive direction are represented schematically below.



Substitution of

$$x(t) = A \cos(\omega t - \delta)$$

into the equation of motion, or equivalently use of complex amplitudes, gives

$$(k - m\omega^2 + i\gamma\omega) \tilde{x} = F_0.$$

Therefore,

$$A(\omega) = |\tilde{x}| = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \gamma^2\omega^2}},$$

and the phase lag satisfies

$$\tan \delta = \frac{\gamma\omega}{k - m\omega^2}.$$

1. For displacement resonance, $A(\omega)$ is maximized. Since F_0 is fixed, it is sufficient to minimize

$$D(\omega) = (k - m\omega^2)^2 + \gamma^2\omega^2.$$

Differentiating,

$$\frac{dD}{d\omega} = 2(k - m\omega^2)(-2m\omega) + 2\gamma^2\omega.$$

For the non-zero resonant frequency,

$$-4m(k - m\omega_r^2) + 2\gamma^2 = 0,$$

so that

$$k - m\omega_r^2 = \frac{\gamma^2}{2m}.$$

Hence

$$\omega_r^2 = \frac{k}{m} - \frac{\gamma^2}{2m^2} = \omega_0^2 - 2\beta^2.$$

For the given data,

$$\omega_0 = \sqrt{\frac{100 \text{ N m}^{-1}}{0.25 \text{ kg}}} = \sqrt{400 \text{ s}^{-2}} = 20.0 \text{ rad s}^{-1},$$

and

$$\beta = \frac{1 \text{ N s m}^{-1}}{2(0.25 \text{ kg})} = 2.00 \text{ s}^{-1}.$$

Thus

$$\omega_r = \sqrt{(20.0)^2 - 2(2.00)^2} \text{ rad s}^{-1} = \sqrt{392} \text{ rad s}^{-1} = 19.8 \text{ rad s}^{-1}.$$

At this frequency,

$$k - m\omega_r^2 = \frac{\gamma^2}{2m} = \frac{(1 \text{ N s m}^{-1})^2}{2(0.25 \text{ kg})} = 2.00 \text{ N m}^{-1}.$$

Therefore,

$$\begin{aligned} A_r &= \frac{F_0}{\sqrt{(k - m\omega_r^2)^2 + \gamma^2\omega_r^2}} \\ &= \frac{5 \text{ N}}{\sqrt{(2.00 \text{ N m}^{-1})^2 + (1 \text{ N s m}^{-1})^2(19.8 \text{ s}^{-1})^2}} \\ &= \frac{5 \text{ N}}{\sqrt{4.00 + 392 \text{ N m}^{-1}}} \\ &= \frac{5}{19.9} \text{ m} \\ &= 0.251 \text{ m}. \end{aligned}$$

Equivalently, using the resonance condition,

$$A_r = \frac{F_0}{\gamma\sqrt{\omega_0^2 - \beta^2}} = \frac{F_0}{\gamma\sqrt{\omega_0^2 - \gamma^2/(4m^2)}}.$$

Under the weak-damping approximation $\beta^2 \ll \omega_0^2$,

$$A_r \simeq \frac{F_0}{\gamma\omega_0} = \frac{5 \text{ N}}{(1 \text{ N s m}^{-1})(20.0 \text{ s}^{-1})} = 0.250 \text{ m}.$$

Thus the exact displacement-resonance result and the usual weak-damping result agree to within about 0.5%.

The dimensional check is

$$[\gamma\omega_0] = (\text{N s m}^{-1})(\text{s}^{-1}) = \text{N m}^{-1},$$

and hence

$$\left[\frac{F_0}{\gamma\omega_0} \right] = \frac{\text{N}}{\text{N m}^{-1}} = \text{m},$$

as required for a displacement amplitude.

A useful physical check is the limit $\gamma \rightarrow 0$: the resonant amplitude varies as $1/\gamma$ and therefore becomes unbounded in the ideal undamped steady-state model. This is the expected resonance behaviour. At the actual displacement-resonance frequency,

$$\tan \delta_r = \frac{\gamma\omega_r}{\gamma^2/(2m)} = \frac{2m\omega_r}{\gamma} \simeq 9.90,$$

so

$$\delta_r \simeq 84.2^\circ.$$

The displacement therefore lags the force by nearly 90° , as expected for weak damping near resonance.

2. The quality factor of a weakly damped oscillator is

$$Q = 2\pi \frac{\text{energy stored}}{\text{energy lost per cycle}} = \frac{\omega_0}{2\beta} = \frac{m\omega_0}{\gamma} = \frac{\sqrt{mk}}{\gamma}.$$

Substitution gives

$$\begin{aligned} Q &= \frac{(0.25 \text{ kg})(20.0 \text{ s}^{-1})}{1 \text{ N s m}^{-1}} \\ &= \frac{5.00 \text{ kg s}^{-1}}{1 \text{ kg s}^{-1}} \\ &= 5.00. \end{aligned}$$

Thus Q is dimensionless.

The same result follows from the fractional energy loss. For weak damping, the mechanical energy decays as

$$E(t) \propto e^{-2\beta t}.$$

During one approximately natural period $T_0 = 2\pi/\omega_0$, the small fractional energy loss is

$$\frac{\Delta E}{E} \approx 2\beta T_0 = \frac{4\pi\beta}{\omega_0}.$$

Consequently,

$$Q = 2\pi \frac{E}{\Delta E} \approx \frac{\omega_0}{2\beta} = 5.00,$$

which confirms the result.

The weak-damping condition is also numerically consistent:

$$\begin{aligned} \frac{\beta}{\omega_0} &= \frac{2.00}{20.0} = 0.100 \\ \frac{\gamma}{2\sqrt{mk}} &= \frac{1}{2\sqrt{(0.25)(100)}} = 0.100. \end{aligned}$$

Moreover, $\omega_r^2 = \omega_0^2 - 2\beta^2 > 0$, so a non-zero displacement-resonance peak exists.

CONCLUSION:

For the convention

$$m\ddot{x} + \gamma\dot{x} + kx = 5 \cos \omega t,$$

the displacement-resonance frequency is

$$\omega_r = 19.8 \text{ rad s}^{-1}.$$

The corresponding steady-state amplitude along the spring axis is

$$A_r = 0.251 \text{ m},$$

or, within the standard weak-damping approximation,

$$A_r \approx \frac{F_0}{\gamma\omega_0} = 0.250 \text{ m}.$$

The displacement lags the driving force by approximately

$$\delta_r \approx 84.2^\circ,$$

which is close to the expected 90° lag at weakly damped resonance.

The oscillator quality factor is

$$Q = \frac{m\omega_0}{\gamma} = \frac{\sqrt{mk}}{\gamma} = 5.00,$$

which is dimensionless. These results apply to the linear steady-state motion with viscous damping, after the transient has decayed. The value $Q = 5$ indicates a moderately sharp resonance, while the conditions $\beta/\omega_0 = 0.10$ and $\omega_r^2 > 0$ confirm that the assumed weakly damped resonance is physically admissible.

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Q6. Consider a damped harmonic oscillator given by the equation $\ddot{x} + 2\mu\dot{x} + \omega_0^2 x = 0$. Obtain the displacements for (i) overdamped, (ii) underdamped and, (iii) critically damped motion.

INTRODUCTION:

The required displacement is obtained by solving the homogeneous, linear equation of motion

$$\ddot{x} + 2\mu\dot{x} + \omega_0^2 x = 0.$$

Here $x(t)$ is the signed displacement from the stable equilibrium position, measured along a fixed x -axis; $x > 0$ and $x < 0$ denote the two opposite directions along this axis. The motion is described in an inertial frame. The parameter $\omega_0 > 0$ is the natural angular frequency of the corresponding undamped oscillator, while $\mu \geq 0$ is the damping constant per unit mass. Thus

$$[\mu] = [\omega_0] = T^{-1}.$$

The equation assumes linear restoring force, viscous damping proportional to velocity, constant coefficients, and the absence of an external driving force.

For a mechanical oscillator of mass m , spring constant k , and viscous coefficient b ,

$$\omega_0^2 = \frac{k}{m}, \quad 2\mu = \frac{b}{m},$$

so the three regimes are distinguished by comparing the damping rate μ with the natural frequency ω_0 . For completeness, let

$$x(0) = x_0, \quad \dot{x}(0) = v_0,$$

where x_0 and v_0 are arbitrary initial displacement and velocity. The general solutions and their corresponding initial-value forms will both be obtained.

SOLUTION:

Since the coefficients are constant, take the trial displacement

$$x(t) = e^{\lambda t}.$$

Then

$$\dot{x} = \lambda e^{\lambda t}, \quad \ddot{x} = \lambda^2 e^{\lambda t}.$$

Substitution into the equation of motion gives

$$(\lambda^2 + 2\mu\lambda + \omega_0^2) e^{\lambda t} = 0.$$

Because $e^{\lambda t} \neq 0$, the characteristic equation is

$$\lambda^2 + 2\mu\lambda + \omega_0^2 = 0,$$

whose roots are

$$\lambda_{1,2} = -\mu \pm \sqrt{\mu^2 - \omega_0^2}.$$

The nature of these roots determines the form of the displacement.

- (i) For overdamped motion, $\mu > \omega_0$.

Define

$$\alpha = \sqrt{\mu^2 - \omega_0^2} > 0.$$

The two characteristic roots are real and distinct:

$$\lambda_1 = -\mu + \alpha, \quad \lambda_2 = -\mu - \alpha.$$

Since $\omega_0 > 0$,

$$\alpha = \sqrt{\mu^2 - \omega_0^2} < \mu,$$

and therefore both roots are negative:

$$\lambda_1 < 0, \quad \lambda_2 < 0.$$

Hence the general displacement is

$$x(t) = C_1 e^{(-\mu+\alpha)t} + C_2 e^{(-\mu-\alpha)t}$$

$$\alpha = \sqrt{\mu^2 - \omega_0^2}.$$

Equivalently,

$$x(t) = e^{-\mu t} (C_1 e^{\alpha t} + C_2 e^{-\alpha t}) = e^{-\mu t} (A \cosh \alpha t + B \sinh \alpha t).$$

To express the constants in terms of x_0 and v_0 , use

$$x(0) = A = x_0.$$

Differentiating,

$$\dot{x} = e^{-\mu t} [-\mu (A \cosh \alpha t + B \sinh \alpha t) + \alpha (A \sinh \alpha t + B \cosh \alpha t)].$$

At $t = 0$,

$$v_0 = -\mu A + \alpha B.$$

Thus

$$B = \frac{v_0 + \mu x_0}{\alpha},$$

and the initial-value form is

$$x(t) = e^{-\mu t} \left[x_0 \cosh(\alpha t) + \frac{v_0 + \mu x_0}{\alpha} \sinh(\alpha t) \right], \quad \alpha = \sqrt{\mu^2 - \omega_0^2}.$$

No sinusoidal factor occurs, so there is no sustained oscillation. Depending on the initial conditions, the oscillator can approach equilibrium from one side or cross the equilibrium at most once before approaching it asymptotically. At sufficiently large t , the more slowly decaying mode dominates:

$$x(t) \sim C_1 e^{-(\mu-\alpha)t}.$$

In the strong-damping limit $\mu \gg \omega_0$,

$$\alpha = \mu \sqrt{1 - \frac{\omega_0^2}{\mu^2}} \approx \mu - \frac{\omega_0^2}{2\mu},$$

so

$$\mu - \alpha \approx \frac{\omega_0^2}{2\mu}.$$

Consequently, the slow part of the displacement behaves as

$$x(t) \sim C_1 \exp\left(-\frac{\omega_0^2}{2\mu} t\right),$$

showing that very large damping can make the final return to equilibrium slow.

- (ii) For underdamped motion, $0 \leq \mu < \omega_0$.

Define the damped angular frequency

$$\omega_d = \sqrt{\omega_0^2 - \mu^2} > 0.$$

Then

$$\sqrt{\mu^2 - \omega_0^2} = i\omega_d,$$

and the characteristic roots are the complex conjugate pair

$$\lambda_{1,2} = -\mu \pm i\omega_d.$$

The corresponding real general displacement is

$$x(t) = e^{-\mu t} (C_1 \cos \omega_d t + C_2 \sin \omega_d t), \quad \omega_d = \sqrt{\omega_0^2 - \mu^2}.$$

It may also be written as

$$x(t) = A_0 e^{-\mu t} \cos(\omega_d t - \phi),$$

where A_0 and ϕ are determined by the initial conditions.

Using $x(0) = x_0$ gives

$$C_1 = x_0.$$

Differentiating,

$$\dot{x} = e^{-\mu t} [-\mu(C_1 \cos \omega_d t + C_2 \sin \omega_d t) - \omega_d C_1 \sin \omega_d t + \omega_d C_2 \cos \omega_d t].$$

At $t = 0$,

$$v_0 = -\mu C_1 + \omega_d C_2,$$

and hence

$$C_2 = \frac{v_0 + \mu x_0}{\omega_d}.$$

Therefore,

$$x(t) = e^{-\mu t} \left[x_0 \cos(\omega_d t) + \frac{v_0 + \mu x_0}{\omega_d} \sin(\omega_d t) \right].$$

The motion is oscillatory, but its amplitude is bounded by an exponentially decreasing envelope proportional to

$$e^{-\mu t}.$$

The damped period is

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{\omega_0^2 - \mu^2}}.$$

Since $\omega_d < \omega_0$, damping reduces the oscillation frequency and increases the period:

$$T_d > \frac{2\pi}{\omega_0}.$$

Successive extrema of the same sign, separated by one damped period, have amplitudes in the ratio

$$\frac{|x(t + T_d)|}{|x(t)|} = e^{-\mu T_d},$$

so the logarithmic decrement is

$$\ln \frac{|x(t)|}{|x(t + T_d)|} = \mu T_d = \frac{2\pi\mu}{\omega_d}.$$

The undamped limit supplies an immediate check. When $\mu \rightarrow 0$,

$$\omega_d \rightarrow \omega_0, \quad e^{-\mu t} \rightarrow 1,$$

and the displacement becomes

$$x(t) \rightarrow x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t,$$

which is the standard displacement of an undamped simple harmonic oscillator.

(iii) For critically damped motion, $\mu = \omega_0$.

The characteristic equation becomes

$$\lambda^2 + 2\omega_0\lambda + \omega_0^2 = 0,$$

or

$$(\lambda + \omega_0)^2 = 0.$$

Thus there is a repeated root

$$\lambda = -\omega_0 = -\mu.$$

For a repeated root of a second-order linear differential equation, the two linearly independent solutions are

$$e^{-\mu t} \quad \text{and} \quad te^{-\mu t}.$$

Hence the general displacement is

$$x(t) = (C_1 + C_2 t)e^{-\mu t}, \quad \mu = \omega_0.$$

Applying the initial conditions,

$$x(0) = C_1 = x_0.$$

Differentiation gives

$$\dot{x} = [C_2 - \mu(C_1 + C_2 t)]e^{-\mu t}.$$

Therefore,

$$v_0 = \dot{x}(0) = C_2 - \mu C_1,$$

so

$$C_2 = v_0 + \mu x_0.$$

The displacement is consequently

$$x(t) = [x_0 + (v_0 + \mu x_0)t]e^{-\mu t}, \quad \mu = \omega_0.$$

This motion is non-oscillatory. Critical damping marks the boundary between oscillatory and non-oscillatory motion and gives the fastest non-oscillatory return to equilibrium for the usual release conditions.

The critically damped expression can also be checked as the continuous limit of the underdamped solution. With fixed x_0 and v_0 ,

$$x(t) = e^{-\mu t} \left[x_0 \cos(\omega_d t) + (v_0 + \mu x_0) \frac{\sin(\omega_d t)}{\omega_d} \right].$$

As $\mu \rightarrow \omega_0^-$, one has $\omega_d \rightarrow 0$, and

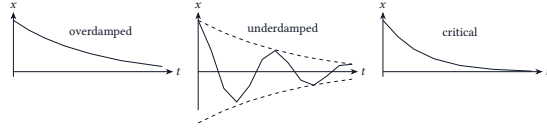
$$\lim_{\omega_d \rightarrow 0} \cos(\omega_d t) = 1, \quad \lim_{\omega_d \rightarrow 0} \frac{\sin(\omega_d t)}{\omega_d} = t.$$

Therefore,

$$x(t) \rightarrow [x_0 + (v_0 + \mu x_0)t]e^{-\mu t},$$

which is exactly the critically damped result.

The characteristic time dependence in the three regimes may be represented schematically as follows; the curves correspond to representative initial conditions and illustrate only the qualitative distinction between monotonic decay and decaying oscillation:



A general physical check follows from the mechanical energy

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2.$$

Its time derivative is

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + m\omega_0^2 x\dot{x} = m\dot{x}(\ddot{x} + \omega_0^2 x).$$

Using the equation of motion,

$$\ddot{x} + \omega_0^2 x = -2\mu\dot{x},$$

gives

$$\frac{dE}{dt} = -2m\mu\dot{x}^2 \leq 0.$$

Thus the damping force continuously removes mechanical energy in all three regimes. For $\mu > 0$, every solution tends to

$$x(t) \rightarrow 0, \quad \dot{x}(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

in agreement with the oscillator ultimately reaching equilibrium.

All exponents are dimensionless because μt , αt , and $\omega_d t$ are dimensionless. In each initial-value formula, the coefficients multiplying the trigonometric or hyperbolic functions have dimensions of displacement:

$$\left[\frac{v_0 + \mu x_0}{\alpha} \right] = \left[\frac{v_0 + \mu x_0}{\omega_d} \right] = L,$$

and

$$[(v_0 + \mu x_0)t] = L.$$

The solutions are therefore dimensionally consistent.

CONCLUSION:

For the signed displacement $x(t)$, with $[\mu] = [\omega_0] = \text{s}^{-1}$, the three damping regimes are:

$\mu > \omega_0$ (overdamped): $x(t) = C_1 e^{[-\mu + \sqrt{\mu^2 - \omega_0^2}]t} + C_2 e^{[-\mu - \sqrt{\mu^2 - \omega_0^2}]t},$
$0 \leq \mu < \omega_0$ (underdamped): $x(t) = e^{-\mu t} \left[C_1 \cos\left(\sqrt{\omega_0^2 - \mu^2} t\right) + C_2 \sin\left(\sqrt{\omega_0^2 - \mu^2} t\right) \right],$
$\mu = \omega_0$ (critically damped): $x(t) = (C_1 + C_2 t) e^{-\mu t}.$

For specified initial displacement x_0 in metres and initial velocity v_0 in metres per second, these become

$x_{\text{over}}(t) = e^{-\mu t} \left[x_0 \cosh(\alpha t) + \frac{v_0 + \mu x_0}{\alpha} \sinh(\alpha t) \right], \quad \alpha = \sqrt{\mu^2 - \omega_0^2},$
$x_{\text{under}}(t) = e^{-\mu t} \left[x_0 \cos(\omega_d t) + \frac{v_0 + \mu x_0}{\omega_d} \sin(\omega_d t) \right], \quad \omega_d = \sqrt{\omega_0^2 - \mu^2},$
$x_{\text{critical}}(t) = [x_0 + (v_0 + \mu x_0)t] e^{-\mu t}, \quad \mu = \omega_0.$

The underdamped oscillator crosses the equilibrium repeatedly, alternately moving in the positive and negative x -directions, while its amplitude decreases as $e^{-\mu t}$. The overdamped and critically damped oscillators return without sustained oscillation; critical damping is the limiting case separating oscillatory from non-oscillatory behavior and provides the fastest non-oscillatory return under the usual release conditions. In every case with $\mu > 0$, the displacement and velocity tend to zero and the oscillator ultimately comes to rest at its equilibrium position because

$$\frac{dE}{dt} = -2m\mu\dot{x}^2 \leq 0.$$

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Q7. Calculate the group and phase velocities for the wave packet corresponding to a relativistic particle.

INTRODUCTION:

The question requires the phase velocity of the individual matter-wave components and the group velocity of the wave-packet envelope representing a free relativistic particle.

Consider a one-dimensional, narrowly localized wave packet constructed from positive-energy plane waves,

$$\Psi(x, t) = \int A(k) \exp[i\{kx - \omega(k)t\}] dk,$$

where $A(k)$ is sharply peaked about the central wave number k_0 . We choose the $+x$ -axis as the propagation direction; hence $k_0 > 0$ and the particle momentum is $p > 0$. The corresponding results remain valid for signed p and k , with the signs specifying the direction of propagation.

For each plane-wave component, the de Broglie and Planck relations are

$$p = \hbar k, \quad E = \hbar \omega.$$

For a free particle of rest mass m , the relativistic energy-momentum relation is

$$E^2 = p^2 c^2 + m^2 c^4,$$

where the positive-energy branch $E > 0$ is used. The particle velocity in the chosen inertial frame is denoted by v , with

$$p = \gamma mv, \quad E = \gamma mc^2, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

The phase velocity and group velocity are defined respectively by

$$v_{\text{ph}} = \frac{\omega}{k}, \quad v_{\text{g}} = \left. \frac{d\omega}{dk} \right|_{k_0}.$$

The identification of v_{g} with the particle velocity assumes a free particle, a sufficiently narrow distribution in k , and a dispersion relation that varies smoothly over the significant spectral width of the packet.

SOLUTION:

Using $E = \hbar\omega$ and $p = \hbar k$, the relativistic dispersion relation is

$$\hbar^2 \omega^2 = \hbar^2 k^2 c^2 + m^2 c^4.$$

Thus, for positive frequency,

$$\omega(k) = \sqrt{k^2 c^2 + \frac{m^2 c^4}{\hbar^2}}.$$

The phase velocity of a component with wave number $k \neq 0$ is

$$v_{\text{ph}} = \frac{\omega}{k} = \frac{E/\hbar}{p/\hbar} = \frac{E}{p}.$$

From the relativistic expressions for energy and momentum,

$$\frac{E}{p} = \frac{\gamma mc^2}{\gamma mv} = \frac{c^2}{v}.$$

Therefore,

$$v_{\text{ph}} = \frac{E}{p} = \frac{c^2}{v}.$$

For a massive particle, $0 < |v| < c$, so

$$|v_{\text{ph}}| = \frac{c^2}{|v|} > c.$$

This superluminal phase velocity does not represent the transport of a localized particle, energy, or information; it describes the motion of surfaces of constant phase of a monochromatic component.

To find the group velocity, consider two neighboring components of the packet,

$$\Psi_1 = \cos(kx - \omega t)$$

$$\Psi_2 = \cos[(k + \Delta k)x - (\omega + \Delta\omega)t].$$

Their superposition contains a slowly varying envelope whose constant-phase condition is

$$\Delta k x - \Delta\omega t = \text{constant}.$$

Differentiating this condition gives the envelope velocity

$$\frac{dx}{dt} = \frac{\Delta\omega}{\Delta k}.$$

In the narrow-packet limit $\Delta k \rightarrow 0$,

$$v_{\text{g}} = \frac{d\omega}{dk}.$$

Since

$$E = \hbar\omega, \quad p = \hbar k,$$

the chain rule gives

$$v_{\text{g}} = \frac{d(E/\hbar)}{d(p/\hbar)} = \frac{dE}{dp}.$$

Differentiating

$$E = \sqrt{p^2 c^2 + m^2 c^4}$$

with respect to p , one obtains

$$\frac{dE}{dp} = \frac{1}{2} (p^2 c^2 + m^2 c^4)^{-1/2} (2pc^2) = \frac{pc^2}{E}.$$

Hence

$$v_{\text{g}} = \frac{pc^2}{E}.$$

Using $p = \gamma mv$ and $E = \gamma mc^2$,

$$v_{\text{g}} = \frac{(\gamma mv)c^2}{\gamma mc^2} = v.$$

Therefore,

$$v_{\text{g}} = v,$$

so the wave-packet envelope propagates with the classical relativistic velocity of the particle.

The two velocities obey

$$v_{\text{ph}} v_{\text{g}} = \left(\frac{E}{p} \right) \left(\frac{pc^2}{E} \right) = c^2,$$

or

$$v_{\text{ph}} v_{\text{g}} = c^2.$$

Several checks establish the consistency of these results.

The dimensions are

$$\left[\frac{E}{p} \right] = \frac{\text{kg m}^2 \text{s}^{-2}}{\text{kg m s}^{-1}} = \text{m s}^{-1},$$

and

$$\left[\frac{pc^2}{E} \right] = \frac{(\text{kg m s}^{-1})(\text{m}^2 \text{s}^{-2})}{\text{kg m}^2 \text{s}^{-2}} = \text{m s}^{-1},$$

so both expressions have the dimensions of velocity.

In the non-relativistic limit $p \ll mc$,

$$E = \sqrt{m^2 c^4 + p^2 c^2} \approx mc^2 + \frac{p^2}{2m}.$$

Consequently,

$$v_g = \frac{dE}{dp} \approx \frac{p}{m} = v \ll c,$$

whereas, if the full relativistic frequency including the rest-energy term is used,

$$v_{ph} = \frac{E}{p} \approx \frac{mc^2}{p} + \frac{p}{2m} \approx \frac{c^2}{v} \gg c.$$

Thus the envelope moves slowly with the particle, while the carrier phase moves rapidly, as expected.

In the ultra-relativistic limit $p \gg mc$,

$$E \approx pc,$$

and hence

$$v_g \approx c, \quad v_{ph} \approx c.$$

For an exactly massless particle, $E = |p|c$, so both velocities have magnitude c . For a massive particle at $p = 0$, the group velocity is zero, while $v_{ph} = \omega/k$ is not defined at $k = 0$, because there is no spatially propagating carrier phase at that point.

For signed momentum, the positive-energy relation gives

$$v_g = \frac{pc^2}{E}, \quad v_{ph} = \frac{E}{p}.$$

Both velocities therefore have the sign of p , so both reverse direction when the momentum reverses, while their product remains c^2 .

CONCLUSION:

For a free relativistic particle with

$$E = \sqrt{p^2 c^2 + m^2 c^4}, \quad p = \hbar k, \quad E = \hbar \omega,$$

the phase and group velocities are

$$v_{ph} = \frac{\omega}{k} = \frac{E}{p} = \frac{c^2}{v},$$

and

$$v_g = \frac{d\omega}{dk} = \frac{dE}{dp} = \frac{pc^2}{E} = v.$$

Both are measured in m s^{-1} , and their signs give the direction of propagation relative to the chosen $+x$ -axis. They satisfy

$$v_{ph} v_g = c^2.$$

For a massive particle, $|v_g| = |v| < c$ while $|v_{ph}| > c$. The superluminal phase velocity does not violate relativity because it does not transport the particle, energy, or information. Under the free-particle and narrow-wave-packet assumptions, it is the group velocity that gives the physical propagation velocity of the relativistic particle. In the massless or ultra-relativistic limit, both velocity magnitudes approach c .

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Q8. Using Huygens' principle for a plane wave travelling from rarer medium 1 to a denser medium 2, show that $\frac{\sin i}{\sin r} = \frac{v_1}{v_2} = \frac{\mu_2}{\mu_1}$, where i and r are the angles of incidence and refraction, respectively. v_1, μ_1 and v_2, μ_2 are the velocities and refractive indices in media 1 and 2, respectively.

INTRODUCTION:

The required result is Snell's law obtained from the Huygens construction of the refracted wavefront. According to Huygens' principle, every point of a wavefront acts as a source of secondary wavelets, and the forward envelope of these wavelets gives the subsequent wavefront.

Assume that both media are homogeneous, isotropic, transparent, and stationary, and that their refractive indices are positive. The plane interface is taken to be $y = 0$, with medium 1 in $y > 0$ and medium 2 in $y < 0$. The wave travels from medium 1 toward decreasing y . The angles i and r are measured from the normal to the interface. Since medium 2 is optically denser,

$$\mu_2 > \mu_1, \quad v_2 < v_1.$$

At the fixed frequency of the incident wave, the refractive index and phase velocity are related by

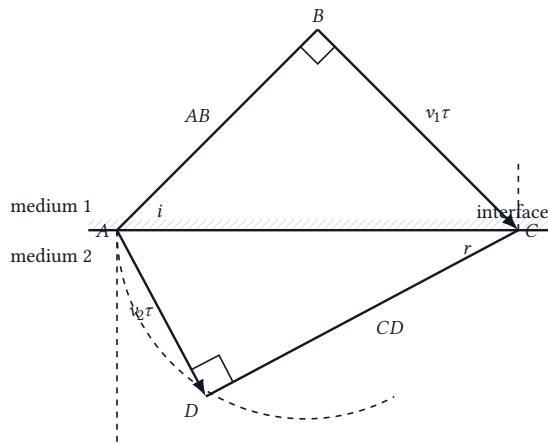
$$\mu_j = \frac{c}{v_j}, \quad j = 1, 2,$$

where c is the speed of light in vacuum.

Let AB be an incident plane wavefront. Point A reaches the interface first. After a time interval τ , point B reaches the interface at C . During the same time, the secondary wavelet emitted from A in medium 2 has radius AD . The tangent CD drawn from C to this wavelet is the refracted wavefront.

SOLUTION:

The Huygens construction is represented schematically below.



The incident ray BC is normal to the incident wavefront AB , so

$$AB \perp BC.$$

The normal to the interface is perpendicular to AC . Hence the angle between the incident ray BC and the interface normal is equal to the angle between AB and AC . Therefore,

$$\angle BAC = i.$$

In the time interval τ , point B of the incident wavefront travels through medium 1 to C . Thus,

$$BC = v_1 \tau.$$

Since $AB \perp BC$, triangle ABC is right-angled at B . Therefore,

$$\sin i = \sin \angle BAC = \frac{BC}{AC} = \frac{v_1 \tau}{AC}.$$

Equivalently,

$$AC = \frac{v_1 \tau}{\sin i}. \quad (1)$$

During precisely the same interval τ , the secondary wavelet originating at A travels through medium 2. Its radius is consequently

$$AD = v_2 \tau.$$

The line CD is tangent to this secondary wavelet at D . A radius drawn to the point of tangency is perpendicular to the tangent, so

$$AD \perp CD.$$

Thus triangle ACD is right-angled at D .

The refracted rays are normal to the refracted wavefront CD . Since the interface normal is perpendicular to AC , the angle between the refracted ray and the interface normal equals the angle between the refracted wavefront CD and the interface AC . Hence,

$$\angle ACD = r.$$

It follows from right triangle ACD that

$$\sin r = \sin \angle ACD = \frac{AD}{AC} = \frac{v_2 \tau}{AC}.$$

Equivalently,

$$AC = \frac{v_2 \tau}{\sin r}. \quad (2)$$

Equating the two expressions for the common distance AC from Eqs. (1) and (2),

$$\frac{v_1 \tau}{\sin i} = \frac{v_2 \tau}{\sin r}.$$

Since $\tau \neq 0$,

$$v_1 \sin r = v_2 \sin i,$$

and hence

$$\boxed{\frac{\sin i}{\sin r} = \frac{v_1}{v_2}}. \quad (3)$$

Using

$$v_1 = \frac{c}{\mu_1}, \quad v_2 = \frac{c}{\mu_2},$$

the velocity ratio becomes

$$\frac{v_1}{v_2} = \frac{c/\mu_1}{c/\mu_2} = \frac{\mu_2}{\mu_1}.$$

Substitution in Eq. (3) gives

$$\boxed{\frac{\sin i}{\sin r} = \frac{v_1}{v_2} = \frac{\mu_2}{\mu_1}}.$$

All three ratios are dimensionless: $\sin i$ and $\sin r$ are dimensionless, v_1/v_2 is a ratio of velocities, and μ_2/μ_1 is a ratio of refractive indices.

As a consistency check, if the two media become optically identical, then

$$\mu_2 = \mu_1 \implies v_2 = v_1 \implies \sin r = \sin i.$$

For $0 \leq i, r < \pi/2$, this gives $r = i$, so there is no bending, as expected. At normal incidence, the limiting form gives $i = r = 0$, and the ray continues along the normal.

For the stated passage from a rarer to a denser medium,

$$\mu_2 > \mu_1 \implies \frac{\mu_2}{\mu_1} > 1.$$

Therefore,

$$\frac{\sin i}{\sin r} > 1 \implies \sin r < \sin i.$$

Since the relevant angles are acute,

$$r < i.$$

Thus the refracted ray bends toward the normal on entering the denser medium.

CONCLUSION:

Huygens' construction gives, for the same propagation time τ ,

$$BC = v_1 \tau, \quad AD = v_2 \tau,$$

and the two right triangles sharing the distance AC give

$$\sin i = \frac{v_1 \tau}{AC}, \quad \sin r = \frac{v_2 \tau}{AC}.$$

Consequently,

$$\boxed{\frac{\sin i}{\sin r} = \frac{v_1}{v_2} = \frac{\mu_2}{\mu_1}}.$$

This result is valid for homogeneous, isotropic media with positive refractive indices and with $v_j = c/\mu_j$ at the common wave frequency. Because medium 2 is denser, $v_2 < v_1$ and $\mu_2 > \mu_1$, so $r < i$: the transmitted ray bends toward the normal.

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Q9. Write equation for damped harmonic oscillations and obtain expression for logarithmic decrement. In a damped harmonic motion, the first amplitude is 10 cm, which reduces to 2 cm after 50 oscillations, each of period 4 seconds. Determine the logarithmic decrement. Also, calculate the number of oscillations in which the amplitude decreases to 25%.

INTRODUCTION:

The problem requires the equation of a freely damped harmonic oscillator, the derivation of its logarithmic decrement, and the use of the observed decay of amplitude to determine both the decrement and the number of oscillations required for a specified reduction.

Let $x(t)$ be the displacement, measured from the equilibrium position along the positive x -axis. For viscous damping, the restoring force and damping force are taken as

$$F_r = -kx, \quad F_d = -c\dot{x},$$

where m is the mass, k is the force constant, and c is the viscous damping coefficient. The damping force is opposite to the instantaneous velocity. No external periodic force is present.

The motion is assumed to be underdamped,

$$c^2 < 4mk,$$

so that the body continues to oscillate while its amplitude decreases exponentially. Define

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad \gamma = \frac{c}{2m}, \quad \omega_d = \sqrt{\omega_0^2 - \gamma^2},$$

where ω_0 is the undamped natural angular frequency, γ is the amplitude-decay constant, and ω_d is the damped angular frequency. The corresponding damped period is

$$T_d = \frac{2\pi}{\omega_d}.$$

The logarithmic decrement δ is defined as the natural logarithm of the ratio of two successive amplitudes measured on the same side of the equilibrium position.

SOLUTION:

1. From Newton's second law,

$$m\ddot{x} = F_r + F_d = -kx - c\dot{x}.$$

Hence the differential equation for free damped harmonic oscillations is

$$\boxed{m\ddot{x} + c\dot{x} + kx = 0}.$$

On division by m ,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0.$$

To solve it, take a trial solution

$$x = e^{st}.$$

Substitution gives the auxiliary equation

$$s^2 + 2\gamma s + \omega_0^2 = 0,$$

whose roots are

$$s = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}.$$

For underdamped motion, $\gamma < \omega_0$, so that

$$s = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2} = -\gamma \pm i\omega_d.$$

Therefore,

$$x(t) = e^{-\gamma t} (C_1 \cos \omega_d t + C_2 \sin \omega_d t).$$

Equivalently,

$$\boxed{x(t) = A_0 e^{-\gamma t} \cos(\omega_d t + \phi)},$$

where A_0 and ϕ are fixed by the initial conditions. Thus the slowly varying amplitude, or envelope, is

$$A(t) = A_0 e^{-\gamma t}.$$

Consider amplitudes A_r and A_{r+1} of two successive maxima on the same side of equilibrium.

These maxima are separated by one damped period T_d . If the first maximum occurs at time t , then

$$A_r = A_0 e^{-\gamma t},$$

and

$$A_{r+1} = A_0 e^{-\gamma(t+T_d)}.$$

Their ratio is therefore

$$\frac{A_r}{A_{r+1}} = \frac{A_0 e^{-\gamma t}}{A_0 e^{-\gamma(t+T_d)}} = e^{\gamma T_d}.$$

By definition, the logarithmic decrement is

$$\delta = \ln \left(\frac{A_r}{A_{r+1}} \right).$$

Consequently,

$$\delta = \gamma T_d.$$

Using $\gamma = c/(2m)$ and $T_d = 2\pi/\omega_d$,

$$\delta = \frac{cT_d}{2m} = \frac{2\pi\gamma}{\omega_d} = \frac{\pi c}{m\sqrt{\omega_0^2 - c^2/(4m^2)}}.$$

More generally, if the amplitudes A_r and A_{r+n} are separated by n complete oscillations, then

$$A_{r+n} = A_r e^{-\gamma n T_d} = A_r e^{-n\delta}.$$

Thus

$$\frac{A_r}{A_{r+n}} = e^{n\delta},$$

and hence

$$\delta = \frac{1}{n} \ln \left(\frac{A_r}{A_{r+n}} \right).$$

The logarithmic decrement is dimensionless, since it is the logarithm of a ratio of two amplitudes. This is also consistent with

$$[\gamma T_d] = s^{-1} s = 1.$$

2. In the given motion,

$$A_1 = 10 \text{ cm}, \quad A_{51} = 2 \text{ cm}, \quad n = 50.$$

The two amplitudes are separated by 50 complete oscillations. Therefore,

$$\delta = \frac{1}{50} \ln \left(\frac{10 \text{ cm}}{2 \text{ cm}} \right).$$

The units cancel within the logarithm:

$$\delta = \frac{1}{50} \ln 5.$$

Using

$$\ln 5 = 1.6094379,$$

one obtains

$$\delta = \frac{1.6094379}{50} = 0.0321888.$$

Hence,

$$\delta \approx 3.22 \times 10^{-2}$$

per oscillation.

Since the period is

$$T_d = 4 \text{ s},$$

the decay constant may also be checked from

$$\gamma = \frac{\delta}{T_d} = \frac{0.0321888}{4 \text{ s}} = 8.0472 \times 10^{-3} \text{ s}^{-1}.$$

Thus the amplitude as a function of time is proportional to

$$A(t) = A_0 e^{-(8.05 \times 10^{-3} \text{ s}^{-1})t}.$$

As a numerical consistency check, the elapsed time in 50 oscillations is

$$t = 50(4 \text{ s}) = 200 \text{ s}.$$

Therefore,

$$\frac{A(200 \text{ s})}{A_0} = e^{-\gamma t} = e^{-(8.0472 \times 10^{-3} \text{ s}^{-1})(200 \text{ s})} = e^{-1.60944} = 0.200.$$

Hence

$$A(200 \text{ s}) = (10 \text{ cm})(0.200) = 2.00 \text{ cm},$$

which reproduces the stated amplitude and verifies the calculation.

3. Let N be the number of oscillations required for the amplitude to become 25% of its initial value. Then

$$A_N = 0.25 A_0.$$

Using

$$A_N = A_0 e^{-N\delta},$$

gives

$$0.25 A_0 = A_0 e^{-N\delta}.$$

After cancelling A_0 ,

$$0.25 = e^{-N\delta}.$$

Taking natural logarithms,

$$\ln(0.25) = -N\delta.$$

Since

$$-\ln(0.25) = \ln 4,$$

the required number is

$$N = \frac{\ln 4}{\delta}.$$

Substituting $\delta = \ln 5/50$,

$$N = \frac{\ln 4}{\ln 5/50} = \frac{50 \ln 4}{\ln 5}.$$

Numerically,

$$N = \frac{50(1.3862944)}{1.6094379} = 43.06.$$

Thus, if a fractional number of periods is permitted,

$$N \simeq 43.1 \text{ oscillations}.$$

The corresponding time is

$$t = NT_d = (43.06)(4 \text{ s}) = 172.2 \text{ s}.$$

If the answer is required in complete oscillations, the amplitude after 43 oscillations is

$$\frac{A_{43}}{A_0} = e^{-43\delta} = e^{-43(0.0321888)} \simeq 0.2506,$$

which is still slightly greater than 25%. After 44 oscillations,

$$\frac{A_{44}}{A_0} = e^{-44\delta} = e^{-44(0.0321888)} \simeq 0.2427,$$

which is below 25%. Hence the amplitude first becomes no more than 25% after

$$44 \text{ complete oscillations}.$$

As a limiting check, when $c \rightarrow 0$, one has $\gamma \rightarrow 0$, so

$$\delta = \gamma T_d \rightarrow 0$$

and $A(t) \rightarrow A_0$. Thus the amplitude becomes constant in the absence of damping, as required by conservation of mechanical energy.

CONCLUSION:

For a freely oscillating mass subject to a linear viscous damping force opposite to its velocity,

$$m\ddot{x} + c\dot{x} + kx = 0.$$

In the underdamped regime $c^2 < 4mk$, its displacement and amplitude are

$$x(t) = A_0 e^{-ct/(2m)} \cos(\omega_d t + \phi)$$

$$A(t) = A_0 e^{-ct/(2m)}$$

where

$$\omega_d = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}.$$

The logarithmic decrement is the dimensionless quantity

$$\delta = \ln \left(\frac{A_r}{A_{r+1}} \right) = \frac{cT_d}{2m} = \frac{2\pi\gamma}{\omega_d},$$

or, for amplitudes separated by n oscillations,

$$\delta = \frac{1}{n} \ln \left(\frac{A_r}{A_{r+n}} \right).$$

For the stated motion,

$$\delta = \frac{\ln 5}{50} \simeq 0.0322 \text{ per oscillation},$$

with the equivalent decay constant

$$\gamma \simeq 8.05 \times 10^{-3} \text{ s}^{-1}.$$

The amplitude falls to exactly 25% of its initial value after

$$N \simeq 43.1 \text{ oscillations}$$

or approximately

$$t \simeq 172 \text{ s}.$$

When only complete oscillations are counted, it first becomes no more than 25% after

$$44 \text{ complete oscillations}.$$

The decreasing positive envelope represents dissipation of mechanical energy by a damping force directed opposite to the velocity; the result is valid for linear viscous damping and underdamped free motion.

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Q10. Find an expression for the average energy of a forced oscillator in the steady-state motion.

INTRODUCTION:

Consider a one-dimensional damped harmonic oscillator of mass m , natural angular frequency ω_0 , and linear damping constant b , driven by a sinusoidal force

$$F(t) = F_0 \cos \omega t.$$

Its equation of motion is

$$m\ddot{x} + b\dot{x} + m\omega_0^2 x = F_0 \cos \omega t.$$

It is convenient to define

$$2\beta = \frac{b}{m},$$

so that

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t.$$

Here ω is the driving angular frequency and β is the damping rate. The displacement x is measured from the equilibrium position, with positive x chosen along the positive direction of the applied force.

After the transient motion has decayed, the oscillator executes steady-state motion at the driving frequency. The required average energy is the time average, over one driving period $T = 2\pi/\omega$, of the instantaneous mechanical energy

$$E(t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2.$$

Because damping continually dissipates energy, $E(t)$ is not constant; the applied force supplies, on average, exactly the energy dissipated during steady-state motion.

SOLUTION:

The steady-state displacement may be written as

$$x(t) = A \cos(\omega t - \delta),$$

where A is the displacement amplitude and δ is the phase lag relative to the applied force. Differentiating,

$$\dot{x}(t) = -A\omega \sin(\omega t - \delta)$$

$$\ddot{x}(t) = -A\omega^2 \cos(\omega t - \delta).$$

To obtain A , it is useful to represent the steady-state motion by

$$x(t) = a \cos \omega t + c \sin \omega t.$$

Then

$$\dot{x} = -a\omega \sin \omega t + c\omega \cos \omega t,$$

and

$$\ddot{x} = -a\omega^2 \cos \omega t - c\omega^2 \sin \omega t.$$

Substitution into the equation of motion gives

$$[(\omega_0^2 - \omega^2)a + 2\beta\omega c] \cos \omega t + [(\omega_0^2 - \omega^2)c - 2\beta\omega a] \sin \omega t = \frac{F_0}{m} \cos \omega t.$$

Equating the coefficients of $\cos \omega t$ and $\sin \omega t$,

$$(\omega_0^2 - \omega^2)a + 2\beta\omega c = \frac{F_0}{m},$$

$$(\omega_0^2 - \omega^2)c - 2\beta\omega a = 0.$$

Squaring and adding the corresponding amplitude relations yields

$$[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2] (a^2 + c^2) = \left(\frac{F_0}{m}\right)^2.$$

Since $A^2 = a^2 + c^2$,

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}.$$

Equivalently, in terms of $b = 2m\beta$,

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}}.$$

The phase lag is determined by

$$\tan \delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2},$$

with the quadrant selected so that $0 \leq \delta \leq \pi$ for positive damping.

The instantaneous kinetic energy is

$$K(t) = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\omega^2 \sin^2(\omega t - \delta),$$

while the instantaneous elastic potential energy is

$$U(t) = \frac{1}{2}m\omega_0^2 x^2 = \frac{1}{2}m\omega_0^2 A^2 \cos^2(\omega t - \delta).$$

The average of a quantity $Q(t)$ over one driving period is

$$\langle Q \rangle = \frac{1}{T} \int_{t_0}^{t_0+T} Q(t) dt.$$

Since

$$\langle \sin^2(\omega t - \delta) \rangle = \langle \cos^2(\omega t - \delta) \rangle = \frac{1}{2},$$

the average kinetic and potential energies are

$$\langle K \rangle = \frac{1}{4}mA^2\omega^2,$$

and

$$\langle U \rangle = \frac{1}{4}mA^2\omega_0^2.$$

Therefore, the average total mechanical energy is

$$\langle E \rangle = \langle K \rangle + \langle U \rangle = \frac{1}{4}mA^2(\omega^2 + \omega_0^2).$$

Substituting the steady-state amplitude,

$$\langle E \rangle = \frac{F_0^2}{4m} \frac{\omega^2 + \omega_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}.$$

In terms of the damping constant b , the same result is

$$\langle E \rangle = \frac{F_0^2 m (\omega^2 + \omega_0^2)}{4 [m^2 (\omega_0^2 - \omega^2)^2 + b^2 \omega^2]}.$$

The result passes the dimensional check. Since

$$[A] = \text{m}, \quad [m(\omega^2 + \omega_0^2)A^2] = \text{kg s}^{-2} \text{m}^2 = \text{J},$$

the expression has the dimensions of energy.

Several physical limits provide useful checks. In the quasistatic limit $\omega \rightarrow 0$,

$$A \rightarrow \frac{F_0}{m\omega_0^2}, \quad \langle K \rangle \rightarrow 0,$$

and hence

$$\langle E \rangle \rightarrow \frac{F_0^2}{4m\omega_0^2}.$$

This is the time-averaged elastic energy associated with the slowly varying displacement

$$x(t) \simeq \frac{F_0}{m\omega_0^2} \cos \omega t.$$

At $\omega = \omega_0$,

$$A = \frac{F_0}{2m\beta\omega_0} = \frac{F_0}{b\omega_0},$$

and the mean kinetic and potential energies become equal:

$$\langle K \rangle = \langle U \rangle.$$

Consequently,

$$\langle E \rangle_{\omega=\omega_0} = \frac{F_0^2}{8m\beta^2} = \frac{mF_0^2}{2b^2}.$$

This finite resonant value requires nonzero damping. In the idealized limit $\beta \rightarrow 0$ at $\omega = \omega_0$, the assumed bounded steady-state motion ceases to exist, and the above steady-state energy diverges, consistently indicating resonant growth.

As a steady-state energy-balance check, the average rate of dissipation is

$$\langle P_{\text{diss}} \rangle = b \langle \dot{x}^2 \rangle = \frac{1}{2} b A^2 \omega^2.$$

The average power delivered by the force is

$$\langle P_{\text{in}} \rangle = \langle F(t) \dot{x}(t) \rangle = \frac{1}{2} F_0 A \omega \sin \delta.$$

Using

$$F_0 \sin \delta = b A \omega,$$

one obtains

$$\langle P_{\text{in}} \rangle = \langle P_{\text{diss}} \rangle,$$

as required for a periodic steady state.

CONCLUSION:

For a linearly damped oscillator driven by $F_0 \cos \omega t$, the steady-state displacement is

$$x(t) = A \cos(\omega t - \delta),$$

where

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

$$\tan \delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

$$2\beta = \frac{b}{m}.$$

Its average kinetic energy, potential energy, and total mechanical energy are, respectively,

$$\langle K \rangle = \frac{1}{4} m A^2 \omega^2, \quad \langle U \rangle = \frac{1}{4} m A^2 \omega_0^2,$$

and

$$\langle E \rangle = \frac{1}{4} m A^2 (\omega^2 + \omega_0^2) = \frac{F_0^2}{4m} \frac{\omega^2 + \omega_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}.$$

The energy is measured in joules and is nonnegative. The result applies after all transient motion has decayed, for sinusoidal forcing and linear viscous damping. At $\omega = \omega_0$, the mean kinetic and potential energies are equal, while nonzero damping keeps the steady-state energy finite. The oscillator does not conserve mechanical energy instantaneously; instead, in steady state, the average power supplied by the driving force equals the average power dissipated by damping.



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