



UPSC PHYSICS OPTIONAL for CSE & IFoS

ABHI PHYSICS

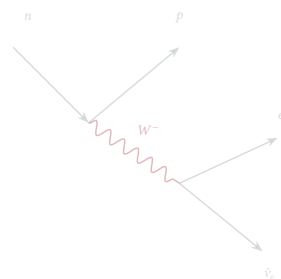
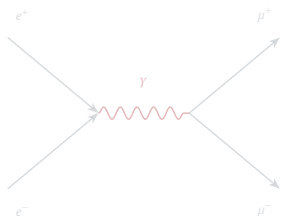
$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$E_n = -\frac{13.6}{n^2} \text{ eV}$$

PREVIOUS YEAR QUESTIONS



41 YEARS QUESTION BANK | 1986-2026

CSE + IFoS

SUBJECT

Mechanics

10 SOLVED QUESTIONS • CSE + IFoS

$$n \rightarrow p + e^- + \bar{\nu}_e$$

$$E^2 = p^2 c^2 + m^2 c^4$$

$$\partial_\mu j^\mu = 0$$

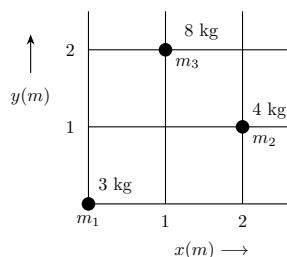
SUBJECT-WISE • TOPIC-WISE • SOURCE-TAGGED

For Civil Services and Indian Forest Service Examinations

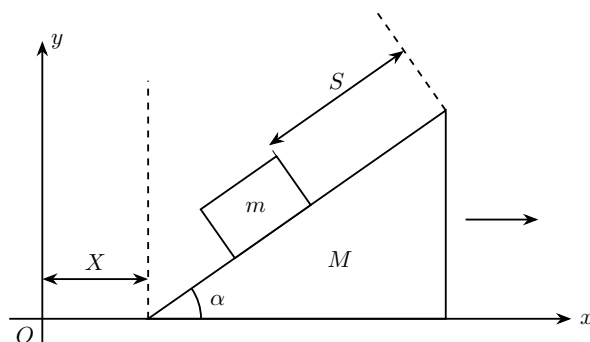
Sample Preview extbar 10 Solved Questions

Question Index

No.	Question	Marks	Year	Exam	Page
1	(i) Define cyclic coordinates and find their connections with the symmetries of the system.; (ii) A system with two degrees of freedom is described by the Lagrangian $L = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 - \frac{\alpha}{q_1}$, α is constant. Find the cyclic coordinates, conserved quantities and symmetries for this system, if any.	10+5=15	2025	IFoS	3
2	A particle of mass m kg having an initial velocity V_0 is subjected to a retarding force proportional to its instantaneous velocity. Obtain the expression for the velocity and position of the particle as a function of time.	10	2024	CSE	6
3	The Lagrangian of a system is given by $L(\vec{r}, \vec{v}) = \frac{1}{2}m(\vec{v} + \vec{a})^2 + \vec{b} \cdot \vec{v} - \vec{c} \cdot \vec{r}$ where, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are constant vectors. Construct the Hamiltonian of the system and find the canonical equations of motion.	8	2024	IFoS	8
4	Construct the Lagrangian of a simple pendulum whose string is replaced by a spring with spring constant k and rest length l_0 oscillating in a vertical $x - y$ plane. Find the equations of motion.	10	2024	IFoS	10
5	A force $\vec{F}$ is given by $\vec{F} = x^2y\hat{x} + zy^2\hat{y} + xz^2\hat{z}$. Determine whether or not the force is conservative.	10	2023	CSE	13
6	What are the coordinates of the centre of mass of the system of masses shown in the figure?	8	2023	IFoS	14



- | | | | | | |
|---|--|----|------|------|----|
| 7 | Consider a particle of mass m moving in a potential field of the form $V(\vec{r}) = V_0(x^2 + y^2)$, where V_0 is a constant. What are the conserved physical quantities for the particle ? | 8 | 2022 | IFoS | 15 |
| 8 | A small block of mass m slides without friction down a wedge-shaped block of mass M and opening angle α as depicted below. | 15 | 2022 | IFoS | 17 |



The wedge-shaped block itself slides along the horizontal frictionless floor along the +ve x -direction. Find the horizontal acceleration $\ddot{X} = d^2X/dt^2$ of the wedge-shaped block by the Lagrangian method and using the displacements X and S of the blocks as generalized coordinates.

- | | | | | | |
|---|---|----|------|-----|----|
| 9 | A body of mass m at rest splits into two masses m_1 and m_2 by an explosion. After the split the bodies move with a total kinetic energy T in opposite direction. Show that their relative speed is $\sqrt{\frac{2Tm}{m_1m_2}}$. | 15 | 2021 | CSE | 18 |
|---|---|----|------|-----|----|

No.	Question	Marks	Year	Exam	Page
10	During take-off, an aircraft accelerates horizontally in a straight line at a rate A . A small bob of mass m is suspended on a string attached to the roof of the cabin, and a hydrogen balloon (total mass m) is tethered to the floor by a string. For each, determine the tension in the string and the equilibrium angle θ between string and vertical. Draw a neat diagram to explain your answer.	10	2021	IFoS	19

IFoS • 2025 • 10+5=15 MARKS

Q1. (i) Define cyclic coordinates and find their connections with the symmetries of the system.; (ii) A system with two degrees of freedom is described by the Lagrangian $L = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2q_1^2\dot{q}_2^2 - \frac{\alpha}{q_1}$, α is constant. Find the cyclic coordinates, conserved quantities and symmetries for this system, if any.

INTRODUCTION:

The problem asks for the relation between cyclic coordinates, continuous symmetries, and conservation laws, and then for its application to the given two-degree-of-freedom system.

Let the generalized coordinates be q_i , with generalized velocities $\dot{q}_i$, and let

$$L = L(q_i, \dot{q}_i, t).$$

A coordinate q_k is called cyclic, ignorable, or redundant when the Lagrangian has no explicit dependence on that coordinate:

$$\frac{\partial L}{\partial q_k} = 0.$$

The Lagrangian may nevertheless depend on the corresponding velocity $\dot{q}_k$. The conjugate generalized momentum is

$$p_k = \frac{\partial L}{\partial \dot{q}_k}.$$

The relevant governing principles are the Euler–Lagrange equations and Noether’s theorem. A cyclic coordinate generates a continuous translation in configuration space,

$$q_k \longrightarrow q_k + \varepsilon, \quad \delta q_k = \varepsilon,$$

where ε is a constant infinitesimal parameter. Since the Lagrangian is independent of q_k , this transformation leaves the Lagrangian and hence the action invariant. Noether’s theorem then identifies p_k as the conserved charge.

For the given system, m_1, m_2 , and α are taken to be constants, $m_1, m_2 > 0$, and the motion is restricted to a regular domain with $q_1 \neq 0$. The positive q_2 -direction is defined as the direction of increasing q_2 ; consequently, the sign of p_2 specifies the sense of motion along that coordinate. No particular physical dimensions are assigned to q_1 and q_2 , although if q_1 is a radial coordinate and q_2 an angle, the q_2 -translation is an ordinary rotation.

SOLUTION:

(i) The Euler–Lagrange equation for q_k is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0.$$

If q_k is cyclic, then

$$\frac{\partial L}{\partial q_k} = 0,$$

and hence

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) = 0.$$

Therefore,

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \text{constant.}$$

The connection with symmetry follows directly. Under the infinitesimal transformation

$$\delta q_i = \varepsilon \delta_{ik}, \quad \delta \dot{q}_i = 0, \quad \delta t = 0,$$

the change in the Lagrangian is

$$\delta L = \sum_i \frac{\partial L}{\partial q_i} \delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i = \varepsilon \frac{\partial L}{\partial q_k} = 0.$$

Thus the action

$$S = \int L dt$$

is invariant under translations along the q_k -direction.

More generally, on using the Euler–Lagrange equations, the variation of L under a continuous coordinate transformation can be written as

$$\delta L = \frac{d}{dt} \left(\sum_i p_i \delta q_i \right).$$

For the cyclic-coordinate transformation $\delta q_k = \varepsilon$, this becomes

$$0 = \delta L = \varepsilon \frac{dp_k}{dt},$$

which again gives

$$\frac{dp_k}{dt} = 0.$$

Hence the chain of implications is

$$\begin{aligned} \frac{\partial L}{\partial q_k} = 0 &\iff q_k \rightarrow q_k + \varepsilon \text{ is a continuous symmetry} \\ &\implies p_k = \text{constant.} \end{aligned}$$

The equivalence here refers to an ordinary constant translation of the coordinate for which the Lagrangian itself is invariant. More general transformations may leave the action invariant up to a total time derivative and lead to the generalized form of Noether’s theorem.

If the Lagrangian has no explicit time dependence,

$$\frac{\partial L}{\partial t} = 0,$$

the system is also invariant under time translations

$$t \longrightarrow t + \varepsilon.$$

The corresponding conserved Noether charge is the energy

$$E = \sum_i \dot{q}_i p_i - L.$$

Time is not one of the configuration coordinates q_i , so this is an additional symmetry rather than an additional cyclic coordinate among the given q_i .

(ii) The Lagrangian is

$$L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 q_1^2 \dot{q}_2^2 - \frac{\alpha}{q_1}.$$

Its explicit coordinate derivatives are

$$\frac{\partial L}{\partial q_1} = m_2 q_1 \dot{q}_2^2 + \frac{\alpha}{q_1^2}, \quad \frac{\partial L}{\partial q_2} = 0.$$

Therefore q_2 is cyclic, whereas q_1 is not cyclic for the stated Lagrangian.

The conjugate momenta are

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = m_1 \dot{q}_1,$$

and

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} = m_2 q_1^2 \dot{q}_2.$$

The Euler–Lagrange equation for q_2 gives

$$\frac{d}{dt} (m_2 q_1^2 \dot{q}_2) = 0.$$

Thus

$$p_2 = m_2 q_1^2 \dot{q}_2 = J = \text{constant},$$

where J denotes the conserved generalized momentum associated with the q_2 -symmetry.

Equivalently,

$$\dot{q}_2 = \frac{J}{m_2 q_1^2}.$$

Thus, when $J > 0$, the motion is in the direction of increasing q_2 ; when $J < 0$, it is in the direction of decreasing q_2 ; and when $J = 0$, q_2 remains constant as long as $q_1 \neq 0$.

The corresponding finite continuous transformation is

$$q'_1 = q_1, \quad q'_2 = q_2 + \varepsilon, \quad t' = t.$$

Since

$$\dot{q}'_1 = \dot{q}_1, \quad \dot{q}'_2 = \dot{q}_2,$$

one has

$$L(q'_1, q'_2, \dot{q}'_1, \dot{q}'_2) = L(q_1, q_2, \dot{q}_1, \dot{q}_2).$$

Therefore the continuous symmetry group contains translations along q_2 . If q_2 is an angular coordinate, this is rotational symmetry, $q_2 \mapsto q_2 + \varepsilon$, and J has the interpretation of the corresponding angular momentum. Without that geometrical interpretation, J should simply be called the conserved generalized momentum conjugate to q_2 .

For completeness, the equation of motion for the non-cyclic coordinate q_1 is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_1} \right) - \frac{\partial L}{\partial q_1} = 0,$$

or

$$m_1 \ddot{q}_1 - m_2 q_1 \dot{q}_2^2 - \frac{\alpha}{q_1^2} = 0.$$

Hence

$$m_1 \ddot{q}_1 = m_2 q_1 \dot{q}_2^2 + \frac{\alpha}{q_1^2}.$$

Substitution of the conserved quantity

$$\dot{q}_2 = \frac{J}{m_2 q_1^2}$$

gives

$$m_1 \ddot{q}_1 = \frac{J^2}{m_2 q_1^3} + \frac{\alpha}{q_1^2}.$$

This explicitly shows that no analogous constant p_1 follows, because the generalized force conjugate to q_1 is generally nonzero.

The Lagrangian also contains no explicit time dependence:

$$\frac{\partial L}{\partial t} = 0.$$

It is therefore invariant under time translations,

$$t \longrightarrow t + \varepsilon,$$

and the energy is conserved. Using

$$E = \dot{q}_1 p_1 + \dot{q}_2 p_2 - L,$$

one obtains

$$\begin{aligned} E &= m_1 \dot{q}_1^2 + m_2 q_1^2 \dot{q}_2^2 - \left(\frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 q_1^2 \dot{q}_2^2 - \frac{\alpha}{q_1} \right) \\ &= \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 q_1^2 \dot{q}_2^2 + \frac{\alpha}{q_1}. \end{aligned}$$

Therefore,

$$E = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2q_1^2\dot{q}_2^2 + \frac{\alpha}{q_1} = \text{constant}.$$

Eliminating $\dot{q}_2$ in favour of J gives the one-dimensional energy form

$$E = \frac{1}{2}m_1\dot{q}_1^2 + \frac{J^2}{2m_2q_1^2} + \frac{\alpha}{q_1}.$$

Thus an effective potential for the q_1 -motion is

$$V_{\text{eff}}(q_1) = \frac{J^2}{2m_2q_1^2} + \frac{\alpha}{q_1}.$$

Indeed,

$$-m_1^{-1} \frac{dV_{\text{eff}}}{dq_1} = \frac{1}{m_1} \left(\frac{J^2}{m_2q_1^3} + \frac{\alpha}{q_1^2} \right),$$

which agrees with the directly derived equation for $\ddot{q}_1$.

The conservation of energy can also be checked explicitly:

$$\begin{aligned} \frac{dE}{dt} &= m_1\dot{q}_1\ddot{q}_1 + m_2q_1\dot{q}_1\dot{q}_2^2 + m_2q_1^2\dot{q}_2\ddot{q}_2 - \frac{\alpha}{q_1^2}\dot{q}_1 \\ &= \dot{q}_1 \left(m_1\ddot{q}_1 - m_2q_1\dot{q}_2^2 - \frac{\alpha}{q_1^2} \right) + \dot{q}_2 \frac{d}{dt} (m_2q_1^2\dot{q}_2) \\ &= 0, \end{aligned}$$

where both Euler–Lagrange equations have been used.

The dimensions are consistent because every term in L and E has the dimensions of energy:

$$[m_1\dot{q}_1^2] = [m_2q_1^2\dot{q}_2^2] = \left[\frac{\alpha}{q_1} \right] = [E].$$

Consequently,

$$[\alpha] = [E][q_1], \quad [J] = [p_2] = \left[\frac{E}{\dot{q}_2} \right].$$

If q_2 is dimensionless, as for an angle measured in radians, then

$$[J] = [E][T],$$

the dimensions of angular momentum. In SI units this is $\text{J s} = \text{kg m}^2 \text{s}^{-1}$, while E is measured in joules.

As a limiting check, if $J = 0$, then

$$\dot{q}_2 = 0, \quad E = \frac{1}{2}m_1\dot{q}_1^2 + \frac{\alpha}{q_1},$$

so the system reduces consistently to one-dimensional q_1 -motion. If q_1 is constant during a motion, conservation of J requires $\dot{q}_2$ to be constant. For $\alpha > 0$, $q_1 > 0$, and $J \neq 0$, both terms on the right-hand side of

$$m_1\ddot{q}_1 = \frac{J^2}{m_2q_1^3} + \frac{\alpha}{q_1^2}$$

are positive, confirming that the acceleration is toward increasing q_1 under the adopted sign convention.

There is also a discrete reflection invariance

$$q_2 \longrightarrow -q_2, \quad \dot{q}_2 \longrightarrow -\dot{q}_2,$$

provided the coordinate domain admits this reflection. It changes J to $-J$ and therefore reverses the sense of motion. Because it is a discrete rather than a continuously parameterized symmetry, it does not by itself produce an additional Noether conserved quantity. Similarly, the velocity-quadratic autonomous Lagrangian is invariant under time reversal, which reverses both generalized momenta but introduces no further independent continuous Noether charge.

CONCLUSION:

A cyclic coordinate is one that is absent explicitly from the Lagrangian,

$$\frac{\partial L}{\partial q_k} = 0.$$

It represents invariance under translations in the corresponding configuration-space direction,

$$q_k \rightarrow q_k + \varepsilon,$$

and its conjugate momentum

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

is conserved.

For the given system, the only cyclic coordinate among q_1 and q_2 is

$$q_2.$$

Its continuous symmetry and conserved generalized momentum are

$$q_2 \rightarrow q_2 + \varepsilon,$$

$$J = p_2 = m_2q_1^2\dot{q}_2 = \text{constant}.$$

The sign of J gives the direction of motion: $J > 0$ corresponds to increasing q_2 , $J < 0$ to decreasing q_2 , and $J = 0$ to fixed q_2 . If q_2 is an angle, this symmetry is rotational invariance and J is angular momentum with SI unit J s .

Since L is autonomous, the system also has time-translation symmetry,

$$t \rightarrow t + \varepsilon,$$

with conserved energy

$$\begin{aligned} E &= \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2q_1^2\dot{q}_2^2 + \frac{\alpha}{q_1} \\ &= \frac{1}{2}m_1\dot{q}_1^2 + \frac{J^2}{2m_2q_1^2} + \frac{\alpha}{q_1} = \text{constant}. \end{aligned}$$

The energy has SI unit joule, and dimensional consistency requires

$$[\alpha] = [E][q_1].$$

The results are valid for constant m_1, m_2, α , with $m_1, m_2 > 0$, on any regular portion of the motion for which $q_1 \neq 0$. The physical inference is that the independence of the dynamics from the absolute value of q_2 makes only changes in q_2 physically relevant and enforces conservation of its conjugate generalized momentum, while invariance under a shift of the time origin enforces conservation of total energy.

CSE • 2024 • 10 MARKS

Q2. A particle of mass m kg having an initial velocity V_0 is subjected to a retarding force proportional to its instantaneous velocity. Obtain the expression for the velocity and position of the particle as a function of time.

INTRODUCTION:

The problem requires the velocity and position of a particle moving under linear resistive damping. Let the motion be one-dimensional along the x -axis, and choose the positive x -direction to be the direction of the initial velocity. Thus,

$$v(0) = V_0 > 0, \quad x(0) = x_0.$$

Since the force is retarding, it acts opposite to the instantaneous velocity. If $k > 0$ is the constant of proportionality, the force law is

$$F_x = -kv,$$

where k is assumed constant and the particle is subject to no other force along the direction of motion. Newton's second law then gives the governing equation

$$m \frac{dv}{dt} = -kv.$$

The minus sign is fixed by the stated coordinate convention: while $v > 0$, the force and acceleration are

in the negative x -direction. The magnitude of the retarding force is kv , so

$$[k] = \frac{[F]}{[v]} = \frac{\text{kg m s}^{-2}}{\text{m s}^{-1}} = \text{kg s}^{-1}.$$

Consequently, m/k has the dimensions of time. Define the relaxation time

$$\tau = \frac{m}{k}.$$

SOLUTION:

Using Newton's second law and the linear retarding-force law,

$$m \frac{dv}{dt} = -kv,$$

or

$$\frac{1}{v} dv = -\frac{k}{m} dt.$$

Integrating from the initial state $(t, v) = (0, V_0)$ to the state $(t, v(t))$,

$$\int_{V_0}^{v(t)} \frac{dv'}{v'} = -\frac{k}{m} \int_0^t dt',$$

which gives

$$\ln \left(\frac{v(t)}{V_0} \right) = -\frac{k}{m} t.$$

Exponentiation therefore yields

$$v(t) = V_0 \exp \left(-\frac{k}{m} t \right) = V_0 e^{-t/\tau}.$$

The acceleration is

$$a(t) = \frac{dv}{dt} = -\frac{kV_0}{m} \exp \left(-\frac{k}{m} t \right).$$

For $V_0 > 0$, this is negative at every finite time, confirming that the force is indeed retarding. Moreover,

$$ma(t) = -kV_0 \exp \left(-\frac{k}{m} t \right) = -kv(t),$$

so the obtained velocity satisfies the original equation of motion.

To determine the position, use

$$\frac{dx}{dt} = v(t) = V_0 \exp \left(-\frac{k}{m} t \right).$$

Integrating from $x(0) = x_0$ to $x(t)$,

$$x(t) - x_0 = \int_0^t V_0 \exp \left(-\frac{k}{m} t' \right) dt'.$$

Since

$$\int \exp \left(-\frac{k}{m} t' \right) dt' = -\frac{m}{k} \exp \left(-\frac{k}{m} t' \right),$$

one obtains

$$x(t) - x_0 = V_0 \left[-\frac{m}{k} \exp\left(-\frac{k}{m}t'\right) \right]_0^t$$

and hence

$$x(t) = x_0 + \frac{mV_0}{k} \left(1 - \exp\left(-\frac{k}{m}t\right) \right) = x_0 + V_0\tau \left(1 - e^{-t/\tau} \right).$$

If the origin is chosen at the initial position, so that $x_0 = 0$, this reduces to

$$x(t) = \frac{mV_0}{k} \left(1 - e^{-kt/m} \right).$$

Differentiating the position expression verifies the kinematic relation:

$$\frac{dx}{dt} = \frac{mV_0}{k} \left(\frac{k}{m} \right) e^{-kt/m} = V_0 e^{-kt/m} = v(t).$$

The dimensions are consistent. The exponential argument is dimensionless because

$$\left[\frac{kt}{m} \right] = \frac{(\text{kg s}^{-1})(\text{s})}{\text{kg}} = 1,$$

and the displacement scale has dimensions

$$\left[\frac{mV_0}{k} \right] = \frac{\text{kg}(\text{m s}^{-1})}{\text{kg s}^{-1}} = \text{m}.$$

For short times $t \ll \tau$, using

$$e^{-t/\tau} = 1 - \frac{t}{\tau} + \frac{t^2}{2\tau^2} + \dots,$$

the velocity and position become

$$v(t) \simeq V_0 \left(1 - \frac{t}{\tau} \right) = V_0 - \frac{kV_0}{m}t,$$

and

$$x(t) \simeq x_0 + V_0t - \frac{kV_0}{2m}t^2.$$

Thus the initial acceleration is

$$a(0) = -\frac{kV_0}{m},$$

as required directly by $ma(0) = -kV_0$.

The zero-damping limit also recovers uniform motion. As $k \rightarrow 0$,

$$v(t) \rightarrow V_0,$$

while

$$\lim_{k \rightarrow 0} \frac{mV_0}{k} \left(1 - e^{-kt/m} \right) = V_0t,$$

so that

$$x(t) \rightarrow x_0 + V_0t.$$

At long times,

$$\lim_{t \rightarrow \infty} v(t) = 0, \quad \lim_{t \rightarrow \infty} x(t) = x_0 + \frac{mV_0}{k}.$$

Therefore, although the particle approaches zero speed only asymptotically, it travels only the finite total distance

$$s_\infty = \frac{mV_0}{k}$$

from its initial position.

As an energy check, the kinetic energy is

$$K(t) = \frac{1}{2}mv^2(t) = \frac{1}{2}mV_0^2 e^{-2kt/m}.$$

Its rate of change is

$$\frac{dK}{dt} = mv \frac{dv}{dt} = v(-kv) = -kv^2 \leq 0.$$

This equals the power delivered by the retarding force,

$$P = F_x v = (-kv)v = -kv^2,$$

confirming that mechanical energy is continuously dissipated and that the sign convention is physically consistent.

CONCLUSION:

For the linear retarding force

$$F_x = -kv, \quad k > 0,$$

with $x(0) = x_0$ and $v(0) = V_0$, the velocity and position are

$$v(t) = V_0 e^{-kt/m} \quad \text{in m s}^{-1},$$

and

$$x(t) = x_0 + \frac{mV_0}{k} \left(1 - e^{-kt/m} \right) \quad \text{in m}.$$

If $x_0 = 0$,

$$x(t) = \frac{mV_0}{k} \left(1 - e^{-kt/m} \right).$$

The characteristic decay time is

$$\tau = \frac{m}{k} \quad \text{in s}.$$

For $V_0 > 0$, the acceleration and retarding force point in the negative x -direction, while the velocity remains in the positive x -direction and decreases exponentially. The motion approaches rest asymptotically, and the total distance travelled is finite:

$$s_\infty = \frac{mV_0}{k} \quad \text{in m}.$$

These results hold when the mass and damping coefficient are constant, the resistive force is exactly proportional to and opposite the instantaneous velocity, and no other force acts along the direction of motion.

IFoS • 2024 • 8 MARKS

Q3. The Lagrangian of a system is given by $L(\vec{r}, \vec{v}) = \frac{1}{2}m(\vec{v} + \vec{a})^2 + \vec{b} \cdot \vec{v} - \vec{c} \cdot \vec{r}$ where, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are constant vectors. Construct the Hamiltonian of the system and find the canonical equations of motion.

INTRODUCTION:

The generalized coordinate is the position vector $\vec{r}$, and the corresponding generalized velocity is

$$\vec{v} = \dot{\vec{r}}.$$

The Hamiltonian is obtained by the Legendre transformation

$$H(\vec{r}, \vec{p}) = \vec{p} \cdot \vec{v} - L(\vec{r}, \vec{v}),$$

where the canonical momentum is

$$\vec{p} = \frac{\partial L}{\partial \vec{v}}.$$

Since $\vec{a}$, $\vec{b}$, and $\vec{c}$ are constant vectors and $m \neq 0$, the relation between $\vec{p}$ and $\vec{v}$ is invertible. Thus the Lagrangian is regular and a Hamiltonian formulation exists. All vectors and derivatives are taken in a fixed inertial Cartesian frame. The canonical equations are

$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}}, \quad \dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}}.$$

SOLUTION:

Differentiating the Lagrangian with respect to $\vec{v}$, component by component, gives

$$p_i = \frac{\partial L}{\partial v_i} = m(v_i + a_i) + b_i.$$

Hence, in vector form,

$$\vec{p} = m(\vec{v} + \vec{a}) + \vec{b}.$$

The Hessian with respect to the velocity components is

$$\frac{\partial^2 L}{\partial v_i \partial v_j} = m\delta_{ij}, \quad \det\left(\frac{\partial^2 L}{\partial v_i \partial v_j}\right) = m^3 \neq 0,$$

which explicitly verifies that the velocity can be expressed uniquely in terms of the canonical momentum.

Solving the momentum relation for $\vec{v}$,

$$m(\vec{v} + \vec{a}) = \vec{p} - \vec{b},$$

so that

$$\vec{v} + \vec{a} = \frac{\vec{p} - \vec{b}}{m}, \quad \vec{v} = \frac{\vec{p} - \vec{b}}{m} - \vec{a}.$$

The Hamiltonian is

$$H = \vec{p} \cdot \vec{v} - L.$$

Substituting the Lagrangian,

$$H = \vec{p} \cdot \vec{v} - \frac{1}{2}m(\vec{v} + \vec{a})^2 - \vec{b} \cdot \vec{v} + \vec{c} \cdot \vec{r}.$$

Combining the terms linear in $\vec{v}$,

$$H = (\vec{p} - \vec{b}) \cdot \vec{v} - \frac{1}{2}m(\vec{v} + \vec{a})^2 + \vec{c} \cdot \vec{r}.$$

Using

$$\vec{p} - \vec{b} = m(\vec{v} + \vec{a}), \quad \vec{v} = \frac{\vec{p} - \vec{b}}{m} - \vec{a},$$

the first term becomes

$$(\vec{p} - \vec{b}) \cdot \vec{v} = (\vec{p} - \vec{b}) \cdot \left(\frac{\vec{p} - \vec{b}}{m} - \vec{a} \right) = \frac{(\vec{p} - \vec{b})^2}{m} - \vec{a} \cdot (\vec{p} - \vec{b}),$$

while

$$\frac{1}{2}m(\vec{v} + \vec{a})^2 = \frac{1}{2}m \left(\frac{\vec{p} - \vec{b}}{m} \right)^2 = \frac{(\vec{p} - \vec{b})^2}{2m}.$$

Therefore,

$$H(\vec{r}, \vec{p}) = \frac{(\vec{p} - \vec{b})^2}{2m} - \vec{a} \cdot (\vec{p} - \vec{b}) + \vec{c} \cdot \vec{r}.$$

Equivalently, on completing the square,

$$H(\vec{r}, \vec{p}) = \frac{(\vec{p} - \vec{b} - m\vec{a})^2}{2m} + \vec{c} \cdot \vec{r} - \frac{1}{2}m\vec{a}^2.$$

The two forms are identical; the final constant term in the second form does not affect Hamilton's equations.

For the first canonical equation,

$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}}.$$

Since

$$\frac{\partial}{\partial \vec{p}} \left[\frac{(\vec{p} - \vec{b})^2}{2m} \right] = \frac{\vec{p} - \vec{b}}{m}, \quad \frac{\partial}{\partial \vec{p}} [-\vec{a} \cdot (\vec{p} - \vec{b})] = -\vec{a},$$

one obtains

$$\dot{\vec{r}} = \frac{\vec{p} - \vec{b}}{m} - \vec{a}.$$

This agrees with the previously inverted momentum–velocity relation and therefore provides an internal consistency check.

For the second canonical equation,

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}}.$$

Only the term $\vec{c} \cdot \vec{r}$ depends on $\vec{r}$, and hence

$$\frac{\partial H}{\partial \vec{r}} = \vec{c}.$$

Therefore,

$$\dot{\vec{p}} = -\vec{c}.$$

The physical equation of motion follows by differentiating the first canonical equation. Since $\vec{a}$ and $\vec{b}$ are constant,

$$\ddot{\vec{r}} = \frac{\dot{\vec{p}}}{m}.$$

Using $\dot{\vec{p}} = -\vec{c}$,

$$m\ddot{\vec{r}} = -\vec{c}.$$

Thus the system describes a particle subjected to the constant force

$$\vec{F} = -\vec{c}.$$

For initial data

$$\vec{r}(0) = \vec{r}_0, \quad \vec{p}(0) = \vec{p}_0,$$

the canonical equations can also be integrated explicitly:

$$\vec{p}(t) = \vec{p}_0 - \vec{c}t,$$

and

$$\dot{\vec{r}}(t) = \frac{\vec{p}_0 - \vec{b}}{m} - \vec{a} - \frac{\vec{c}}{m}t.$$

Therefore,

$$\vec{r}(t) = \vec{r}_0 + \left(\frac{\vec{p}_0 - \vec{b}}{m} - \vec{a} \right)t - \frac{\vec{c}}{2m}t^2.$$

The dimensions are consistent:

$$[\vec{p}] = [\vec{b}] = MLT^{-1}$$

$$[\vec{a}] = LT^{-1}$$

$$[\vec{c}] = MLT^{-2}.$$

Consequently,

$$\left[\frac{(\vec{p} - \vec{b})^2}{2m} \right] = [\vec{a} \cdot (\vec{p} - \vec{b})] = [\vec{c} \cdot \vec{r}] = ML^2T^{-2},$$

so every term in H has the dimension of energy.

Because the Lagrangian and Hamiltonian have no explicit time dependence,

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0.$$

The Hamiltonian is therefore conserved along the motion. Indeed, using Hamilton's equations,

$$\frac{dH}{dt} = \frac{\partial H}{\partial \vec{r}} \cdot \dot{\vec{r}} + \frac{\partial H}{\partial \vec{p}} \cdot \dot{\vec{p}} = \vec{c} \cdot \dot{\vec{r}} + \dot{\vec{r}} \cdot (-\vec{c}) = 0.$$

As a limiting check, if

$$\vec{a} = \vec{b} = \vec{0},$$

then

$$L = \frac{1}{2}m\vec{v}^2 - \vec{c} \cdot \vec{r}, \quad H = \frac{\vec{p}^2}{2m} + \vec{c} \cdot \vec{r},$$

and Hamilton's equations reduce to

$$\dot{\vec{r}} = \frac{\vec{p}}{m}, \quad \dot{\vec{p}} = -\vec{c},$$

which are the standard equations for a particle in the linear potential

$$V(\vec{r}) = \vec{c} \cdot \vec{r}.$$

The role of the constant vectors $\vec{a}$ and $\vec{b}$ can also be seen by expanding the Lagrangian:

$$L = \frac{1}{2}m\vec{v}^2 + (m\vec{a} + \vec{b}) \cdot \vec{v} + \frac{1}{2}m\vec{a}^2 - \vec{c} \cdot \vec{r}.$$

Since

$$(m\vec{a} + \vec{b}) \cdot \vec{v} = \frac{d}{dt} [(m\vec{a} + \vec{b}) \cdot \vec{r}],$$

the velocity-linear term is a total time derivative, while $\frac{1}{2}m\vec{a}^2$ is a constant. These terms alter the definition of the canonical momentum and shift the Hamiltonian by canonical-momentum and constant terms, but they do not change the physical equation $m\ddot{\vec{r}} = -\vec{c}$.

CONCLUSION:

The canonical momentum is

$$\vec{p} = m(\dot{\vec{r}} + \vec{a}) + \vec{b}.$$

The Hamiltonian, expressed entirely in terms of the canonical variables $(\vec{r}, \vec{p})$, is

$$H(\vec{r}, \vec{p}) = \frac{(\vec{p} - \vec{b})^2}{2m} - \vec{a} \cdot (\vec{p} - \vec{b}) + \vec{c} \cdot \vec{r}$$

or, equivalently,

$$H(\vec{r}, \vec{p}) = \frac{(\vec{p} - \vec{b} - m\vec{a})^2}{2m} + \vec{c} \cdot \vec{r} - \frac{1}{2}m\vec{a}^2.$$

The canonical equations of motion are

$$\dot{\vec{r}} = \frac{\vec{p} - \vec{b}}{m} - \vec{a}, \quad \dot{\vec{p}} = -\vec{c}.$$

They imply

$$m\ddot{\vec{r}} = -\vec{c},$$

so the particle has the constant acceleration

$$\ddot{\vec{r}} = -\frac{\vec{c}}{m},$$

directed opposite to $\vec{c}$. This result is valid for constant $\vec{a}$, $\vec{b}$, and $\vec{c}$, with $m \neq 0$. The vectors $\vec{a}$ and $\vec{b}$ shift the relation between canonical momentum and mechanical velocity but do not change the physical force, which is $\vec{F} = -\vec{c}$.

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Q4. Construct the Lagrangian of a simple pendulum whose string is replaced by a spring with spring constant k and rest length l_0 oscillating in a vertical $x-y$ plane. Find the equations of motion.

INTRODUCTION:

The system is an elastic pendulum consisting of a point mass m attached to the lower end of a massless Hookean spring. The upper end of the spring is fixed at a pivot O , and the motion is restricted to a vertical plane. The problem requires the Lagrangian and the coupled nonlinear equations governing the radial extension and angular motion.

Choose an inertial Cartesian frame with origin at O , the x -axis horizontal towards the right, and the y -axis vertically upward. Let $r(t) > 0$ be the instantaneous length of the spring, and let $\theta(t)$ be the angle measured from the vertically downward direction towards the positive x -direction. Thus,

$$x = r \sin \theta, \quad y = -r \cos \theta.$$

The generalized coordinates are

$$q_1 = r, \quad q_2 = \theta.$$

The following assumptions are made:

1. The bob is a point mass m , while the spring is massless.
2. The spring remains straight and obeys Hooke's law throughout the motion.
3. The motion is planar, and the pivot is fixed in an inertial frame.
4. Damping and air resistance are absent, so all forces are conservative.
5. The spring length remains positive, $r > 0$.

With the outward radial unit vector directed from the pivot towards the mass, Hooke's law gives

$$\vec{F}_s = -k(r - l_0)\hat{e}_r.$$

Thus, when $r > l_0$, the stretched spring pulls the mass inward.

The governing principle is Hamilton's principle,

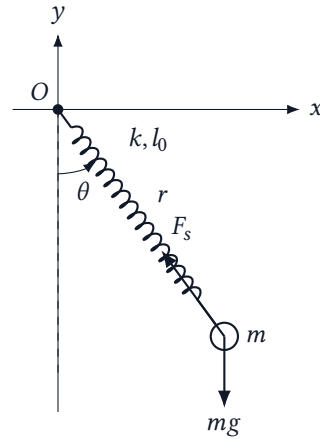
$$\delta \int_{t_1}^{t_2} L dt = 0, \quad L = T - V,$$

which leads to the Euler-Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0.$$

SOLUTION:

The coordinate and sign conventions are represented by the following diagram.



From

$$x = r \sin \theta, \quad y = -r \cos \theta,$$

the velocity components are

$$\dot{x} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta,$$

and

$$\dot{y} = -\dot{r} \cos \theta + r \dot{\theta} \sin \theta.$$

Therefore,

$$\begin{aligned} \dot{x}^2 + \dot{y}^2 &= (\dot{r} \sin \theta + r \dot{\theta} \cos \theta)^2 + (-\dot{r} \cos \theta + r \dot{\theta} \sin \theta)^2 \\ &= \dot{r}^2 (\sin^2 \theta + \cos^2 \theta) + r^2 \dot{\theta}^2 (\sin^2 \theta + \cos^2 \theta) \\ &\quad + 2r\dot{r}\dot{\theta} \sin \theta \cos \theta - 2r\dot{r}\dot{\theta} \sin \theta \cos \theta \\ &= \dot{r}^2 + r^2 \dot{\theta}^2. \end{aligned}$$

Hence the kinetic energy is

$$T = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2).$$

Taking the gravitational potential energy to be zero at the level of the pivot,

$$V_g = mgy = -mgr \cos \theta.$$

The elastic potential energy of the spring is

$$V_s = \frac{k}{2}(r - l_0)^2.$$

Thus,

$$V = V_s + V_g = \frac{k}{2}(r - l_0)^2 - mgr \cos \theta.$$

The Lagrangian is consequently

$$L(r, \theta, \dot{r}, \dot{\theta}) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{2}(r - l_0)^2 + mgr \cos \theta.$$

For the radial coordinate r ,

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = m\ddot{r},$$

and

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - k(r - l_0) + mg \cos \theta.$$

The Euler–Lagrange equation for r is therefore

$$m\ddot{r} - \{mr\dot{\theta}^2 - k(r - l_0) + mg \cos \theta\} = 0.$$

Hence,

$$m(\ddot{r} - r\dot{\theta}^2) = mg \cos \theta - k(r - l_0).$$

Equivalently,

$$\ddot{r} - r\dot{\theta}^2 = g \cos \theta - \frac{k}{m}(r - l_0).$$

This equation has the expected radial-force interpretation. The outward radial component of the acceleration is

$$a_r = \ddot{r} - r\dot{\theta}^2.$$

The component of gravity along the outward radial direction is

$$mg \cos \theta,$$

whereas the spring force is

$$-k(r - l_0).$$

Thus the radial equation is also obtained directly from Newton's second law along $\hat{\mathbf{e}}_r$.

For the angular coordinate θ ,

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}.$$

Therefore,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}.$$

Also,

$$\frac{\partial L}{\partial \theta} = -mgr \sin \theta.$$

The angular Euler–Lagrange equation becomes

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr \sin \theta = 0.$$

For $m > 0$ and $r > 0$, division by mr gives

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + g \sin \theta = 0.$$

Equivalently,

$$\ddot{\theta} + 2\frac{\dot{r}}{r}\dot{\theta} + \frac{g}{r} \sin \theta = 0.$$

The term

$$2\frac{\dot{r}}{r}\dot{\theta}$$

expresses the coupling between radial and angular motion due to the changing distance of the mass from the pivot.

The angular equation can also be checked by taking moments about O . Since the spring force passes through O , it exerts no torque about the pivot. The angular momentum about O is

$$\mathcal{L}_O = mr^2\dot{\theta},$$

while the gravitational torque in the positive- θ convention is

$$\tau_O = -mgr \sin \theta.$$

Thus,

$$\frac{d}{dt} (mr^2\dot{\theta}) = -mgr \sin \theta,$$

which reproduces the angular equation.

For the static downward equilibrium,

$$\theta_{\text{eq}} = 0, \quad \dot{r} = \dot{\theta} = \ddot{r} = \ddot{\theta} = 0.$$

The radial equation then gives

$$0 = mg - k(r_{\text{eq}} - l_0),$$

so that

$$r_{\text{eq}} = l_0 + \frac{mg}{k}.$$

Therefore, gravity produces the physically expected extension

$$r_{\text{eq}} - l_0 = \frac{mg}{k} > 0.$$

At the inverted position $\theta = \pi$,

$$mg \cos \pi = -mg,$$

so gravity has an inward radial component, consistently with the adopted convention.

The dimensional consistency of the radial equation follows from

$$[\ddot{r}] = [r\dot{\theta}^2] = [g] = \text{m s}^{-2},$$

and

$$\left[\frac{k}{m}(r - l_0) \right] = \frac{Nm^{-1}}{kg} m = \frac{kg s^{-2}}{kg} m = m s^{-2}.$$

In the angular equation,

$$[\ddot{\theta}] = \left[\frac{\dot{r}\dot{\theta}}{r} \right] = \left[\frac{g}{r} \right] = s^{-2},$$

because the angle in radians is dimensionless.

Since the Lagrangian has no explicit time dependence, the corresponding energy function

$$E = \dot{r} \frac{\partial L}{\partial \dot{r}} + \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} - L$$

is conserved. Substitution gives

$$E = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{k}{2} (r - l_0)^2 - mgr \cos \theta.$$

This conservation law may be verified directly without assuming the result. Differentiating E gives

$$\begin{aligned} \frac{dE}{dt} &= m\dot{r}\ddot{r} + mr\dot{\theta}^2 + mr^2\ddot{\theta} + k(r - l_0)\dot{r} \\ &\quad - mgr\dot{\cos \theta} + mgr\dot{\sin \theta}. \end{aligned}$$

Regrouping the terms according to the two equations of motion,

$$\begin{aligned} \frac{dE}{dt} &= \dot{r} [m\ddot{r} - mr\dot{\theta}^2 + k(r - l_0) - mg \cos \theta] \\ &\quad + \dot{\theta} [mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr \sin \theta]. \end{aligned}$$

Each square bracket vanishes by the corresponding Euler-Lagrange equation. Therefore,

$$\frac{dE}{dt} = 0,$$

confirming conservation of total mechanical energy.

As a limiting check, suppose the spring is sufficiently stiff that radial departures from equilibrium are negligible. Then

$$r \simeq r_{eq} = \text{constant}, \quad \dot{r} \simeq 0.$$

The angular equation reduces to

$$\ddot{\theta} + \frac{g}{r_{eq}} \sin \theta = 0,$$

which is the nonlinear equation of an ordinary simple pendulum of length r_{eq} . For small angular displacements,

$$\sin \theta \simeq \theta,$$

and hence

$$\ddot{\theta} + \frac{g}{r_{eq}} \theta = 0.$$

Thus,

$$\omega_{\theta} \simeq \sqrt{\frac{g}{r_{eq}}},$$

as expected for a simple pendulum.

For small oscillations about the stable downward equilibrium, write

$$r = r_{eq} + \rho, \quad |\rho| \ll r_{eq}, \quad |\theta| \ll 1.$$

To first order in the small quantities,

$$\sin \theta \simeq \theta, \quad \cos \theta \simeq 1,$$

while

$$r\dot{\theta}^2, \quad \dot{\rho}\dot{\theta}, \quad \rho\dot{\theta}$$

are of second order and are neglected. Using

$$k(r_{eq} - l_0) = mg,$$

the radial equation becomes

$$m\ddot{\rho} = -k\rho,$$

or

$$\ddot{\rho} + \frac{k}{m} \rho = 0.$$

The angular equation becomes

$$\ddot{\theta} + \frac{g}{r_{eq}} \theta = 0.$$

Thus the radial and angular motions decouple to linear order, with frequencies

$$\omega_r = \sqrt{\frac{k}{m}}, \quad \omega_{\theta} = \sqrt{\frac{g}{r_{eq}}} = \sqrt{\frac{g}{l_0 + mg/k}}.$$

CONCLUSION:

For a point mass m suspended from a massless Hookean spring of spring constant k and rest length l_0 , with r measured outward from the pivot and θ measured from the vertically downward direction towards positive x , the Lagrangian is

$$L = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{k}{2} (r - l_0)^2 + mgr \cos \theta.$$

The coupled nonlinear equations of motion are

$$m(\ddot{r} - r\dot{\theta}^2) = mg \cos \theta - k(r - l_0)$$

in the outward radial direction, and

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + g \sin \theta = 0$$

in the direction of increasing θ .

Here r and l_0 are measured in m, m in kg, k in N m^{-1} , θ in radians, and g in m s^{-2} . The stable downward equilibrium is

$$\theta_{\text{eq}} = 0, \quad r_{\text{eq}} = l_0 + \frac{mg}{k},$$

so the equilibrium extension is mg/k in the downward direction.

The conserved mechanical energy is

$$E = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{k}{2} (r - l_0)^2 - mgr \cos \theta = \text{constant}.$$

The equations are valid for planar motion with $r > 0$, a fixed pivot, a point bob, a massless spring obeying Hooke's law, and negligible dissipative forces. In the stiff-spring limit the angular equation reduces to the ordinary simple-pendulum equation. Near the stable equilibrium, the radial and angular small-oscillation frequencies are

$$\omega_r = \sqrt{\frac{k}{m}}, \quad \omega_\theta = \sqrt{\frac{g}{l_0 + mg/k}}.$$

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Q5. A force $\vec{F}$ is given by $\vec{F} = x^2 y \hat{x} + zy^2 \hat{y} + xz^2 \hat{z}$. Determine whether or not the force is conservative.

INTRODUCTION:

The demand is to determine whether the given force field can be expressed as the gradient of a single-valued scalar potential. Write

$$\begin{aligned} \vec{F} &= F_x \hat{x} + F_y \hat{y} + F_z \hat{z} \\ F_x &= x^2 y \\ F_y &= zy^2 \\ F_z &= xz^2. \end{aligned}$$

The field is polynomial and hence continuously differentiable throughout the simply connected domain $\mathbb{R}^3$. Therefore, a necessary and sufficient condition for $\vec{F}$ to be conservative on this domain is

$$\nabla \times \vec{F} = \vec{0}$$

at every point. Equivalently, the relevant mixed partial derivatives of the components must agree. The standard right-handed Cartesian coordinate system is used.

SOLUTION:

The curl of the force field is

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & zy^2 & xz^2 \end{vmatrix}.$$

Expanding the determinant,

$$\nabla \times \vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z}.$$

The required derivatives are

$$\frac{\partial F_z}{\partial y} = \frac{\partial(xz^2)}{\partial y} = 0, \quad \frac{\partial F_y}{\partial z} = \frac{\partial(zy^2)}{\partial z} = y^2,$$

$$\frac{\partial F_x}{\partial z} = \frac{\partial(x^2 y)}{\partial z} = 0, \quad \frac{\partial F_z}{\partial x} = \frac{\partial(xz^2)}{\partial x} = z^2,$$

and

$$\frac{\partial F_y}{\partial x} = \frac{\partial(zy^2)}{\partial x} = 0, \quad \frac{\partial F_x}{\partial y} = \frac{\partial(x^2 y)}{\partial y} = x^2.$$

Hence

$$\nabla \times \vec{F} = -y^2 \hat{x} - z^2 \hat{y} - x^2 \hat{z}.$$

This curl is not identically zero. It vanishes only at the isolated point

$$x = y = z = 0,$$

which is insufficient: conservativeness requires the curl to vanish throughout an open domain.

The conclusion can also be checked directly through the equality of mixed derivatives that would follow from a potential V . If

$$\vec{F} = -\nabla V,$$

then, for example,

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}.$$

But here

$$\frac{\partial F_x}{\partial y} = x^2, \quad \frac{\partial F_y}{\partial x} = 0,$$

which are unequal for $x \neq 0$. Thus no scalar potential can generate the given field throughout $\mathbb{R}^3$.

As an independent circulation check, consider a rectangle in the plane $z = 0$, with

$$a \leq x \leq b, \quad c \leq y \leq d,$$

oriented counterclockwise as viewed from $+\hat{z}$. By Stokes' theorem,

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{z} dA.$$

Since

$$(\nabla \times \vec{F}) \cdot \hat{z} = -x^2,$$

one obtains

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = - \int_c^d \int_a^b x^2 dx dy = - \frac{(d-c)(b^3 - a^3)}{3},$$

which is generally nonzero. This confirms that the work depends on the path and that the work around a closed path need not vanish.

Dimensionally, each component of $\nabla \times \vec{F}$ has the dimensions of force per unit length, as expected after taking one spatial derivative of a force field. For a loop whose dimensions shrink to zero, the circulation also tends to zero with the enclosed area and the local curl; this limiting behavior is consistent with Stokes' theorem but does not make the field conservative on any finite open region where the curl is nonzero.

CONCLUSION:

The given force field is

not conservative

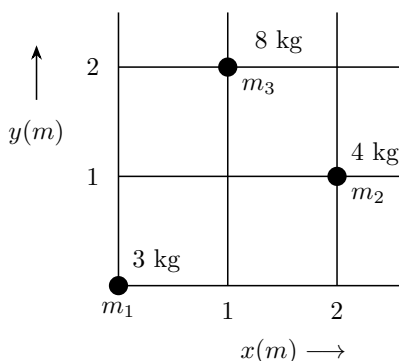
on $\mathbb{R}^3$, because

$$\nabla \times \vec{F} = -y^2 \hat{x} - z^2 \hat{y} - x^2 \hat{z} \neq \vec{0}$$

in general. Consequently, no single-valued scalar potential exists for this force over $\mathbb{R}^3$, the work done is path-dependent, and the circulation around a closed curve can be nonzero.

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Q6. What are the coordinates of the centre of mass of the system of masses shown in the figure?



INTRODUCTION:

The system consists of three point masses in the xy -plane. Taking the origin and the positive x - and y -directions exactly as indicated in the figure, their masses and position vectors are

$$m_1 = 3 \text{ kg}, \quad \vec{r}_1 = (0 \text{ m}, 0 \text{ m}),$$

$$m_2 = 4 \text{ kg}, \quad \vec{r}_2 = (2 \text{ m}, 1 \text{ m}),$$

and

$$m_3 = 8 \text{ kg}, \quad \vec{r}_3 = (1 \text{ m}, 2 \text{ m}).$$

For discrete point masses, the centre-of-mass position vector is the mass-weighted mean

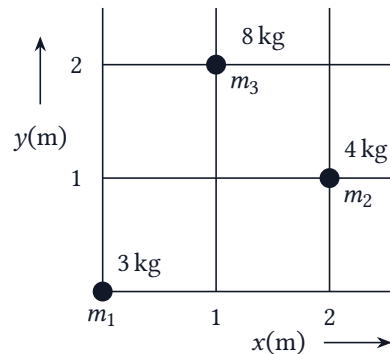
$$\vec{R}_{\text{CM}} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}.$$

Hence its Cartesian coordinates are

$$x_{\text{CM}} = \frac{\sum_i m_i x_i}{\sum_i m_i}, \quad y_{\text{CM}} = \frac{\sum_i m_i y_i}{\sum_i m_i}.$$

SOLUTION:

The configuration represented by the given figure is



The total mass is

$$M = m_1 + m_2 + m_3 = 3 \text{ kg} + 4 \text{ kg} + 8 \text{ kg} = 15 \text{ kg}.$$

The x -coordinate of the centre of mass is

$$\begin{aligned} x_{\text{CM}} &= \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{M} \\ &= \frac{(3 \text{ kg})(0 \text{ m}) + (4 \text{ kg})(2 \text{ m}) + (8 \text{ kg})(1 \text{ m})}{15 \text{ kg}} \\ &= \frac{16 \text{ kg m}}{15 \text{ kg}} = \frac{16}{15} \text{ m} \approx 1.07 \text{ m}. \end{aligned}$$

Similarly, the y -coordinate is

$$\begin{aligned} y_{\text{CM}} &= \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{M} \\ &= \frac{(3 \text{ kg})(0 \text{ m}) + (4 \text{ kg})(1 \text{ m}) + (8 \text{ kg})(2 \text{ m})}{15 \text{ kg}} \\ &= \frac{20 \text{ kg m}}{15 \text{ kg}} = \frac{4}{3} \text{ m} \approx 1.33 \text{ m}. \end{aligned}$$

Thus

$$\vec{R}_{\text{CM}} = \left(\frac{16}{15} \hat{i} + \frac{4}{3} \hat{j} \right) \text{ m}.$$

Both coordinates lie between the minimum and maximum particle coordinates, $0 \leq x_{\text{CM}} \leq 2 \text{ m}$ and $0 \leq$

$y_{CM} \leq 2 \text{ m}$, as required for a positive-mass weighted mean.

CONCLUSION:

With respect to the origin and axes shown in the figure, the coordinates of the centre of mass are

$$(x_{CM}, y_{CM}) = \left(\frac{16}{15}, \frac{4}{3} \right) \text{ m} \approx (1.07, 1.33) \text{ m}.$$

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Q7. Consider a particle of mass m moving in a potential field of the form $V(\vec{r}) = V_0(x^2 + y^2)$, where V_0 is a constant. What are the conserved physical quantities for the particle ?

INTRODUCTION:

The demand is to identify the constants of motion by examining the continuous symmetries of the potential. We use a fixed Cartesian inertial frame with

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}, \quad \rho = \sqrt{x^2 + y^2},$$

and take positive rotations about the z -axis according to the right-hand rule. The potential can then be written as

$$V(\vec{r}) = V_0\rho^2.$$

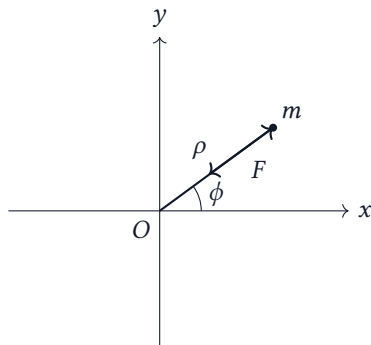
It is independent of time t , the coordinate z , and the azimuthal angle ϕ . By Noether's theorem, these invariances imply conservation of total mechanical energy, linear momentum along the z -axis, and angular momentum about the z -axis, respectively. The motion is assumed to be governed by the non-relativistic Lagrangian

$$\mathcal{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V_0(x^2 + y^2),$$

with m and V_0 constant and with no non-conservative or explicitly time-dependent forces.

SOLUTION:

The geometry and the force directions in the transverse plane are represented by:



Since

$$\nabla V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z} = 2V_0x\hat{x} + 2V_0y\hat{y},$$

the force is

$$\vec{F} = -\nabla V = -2V_0x\hat{x} - 2V_0y\hat{y}, \quad F_z = 0.$$

For $V_0 > 0$, the transverse force points toward the z -axis; for $V_0 < 0$, it points away from that axis. The conservation laws themselves hold for either sign of V_0 .

1. Conservation of total mechanical energy

The potential has no explicit time dependence:

$$\frac{\partial V}{\partial t} = 0.$$

The total mechanical energy is therefore

$$E = T + V = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V_0(x^2 + y^2).$$

To verify its conservation directly,

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + m\dot{y}\ddot{y} + m\dot{z}\ddot{z} + 2V_0x\dot{x} + 2V_0y\dot{y}.$$

The equations of motion are

$$m\ddot{x} = -2V_0x, \quad m\ddot{y} = -2V_0y, \quad m\ddot{z} = 0.$$

Substitution gives

$$\frac{dE}{dt} = \dot{x}(-2V_0x) + \dot{y}(-2V_0y) + 0 + 2V_0x\dot{x} + 2V_0y\dot{y} = 0.$$

Hence

$$E = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + V_0(x^2 + y^2) = \text{constant}.$$

2. Conservation of the z -component of linear momentum

Since V is independent of z , the coordinate z is cyclic:

$$\frac{\partial V}{\partial z} = 0.$$

Consequently,

$$\dot{p}_z = F_z = -\frac{\partial V}{\partial z} = 0,$$

and therefore

$$p_z = m\dot{z} = \text{constant}.$$

Thus the motion parallel to the z -axis is uniform:

$$z(t) = z_0 + \frac{p_z}{m}t.$$

The full linear momentum is not generally conserved, because

$$\dot{p}_x = -2V_0x, \quad \dot{p}_y = -2V_0y$$

are not generally zero.

3. Conservation of angular momentum about the z-axis

The potential depends on x and y only through $x^2 + y^2 = \rho^2$. It is therefore invariant under rotations

$$\begin{aligned}x &\longrightarrow x \cos \alpha - y \sin \alpha \\y &\longrightarrow x \sin \alpha + y \cos \alpha\end{aligned}$$

about the z -axis. The corresponding conserved quantity is

$$L_z = (\vec{r} \times \vec{p})_z = x p_y - y p_x = m(x\dot{y} - y\dot{x}).$$

Direct differentiation yields

$$\frac{dL_z}{dt} = x\dot{p}_y - y\dot{p}_x = x(-2V_0\dot{y}) - y(-2V_0\dot{x}) = 0.$$

Equivalently, the z -component of the torque is

$$\tau_z = (\vec{r} \times \vec{F})_z = xF_y - yF_x = 0.$$

Thus

$$L_z = x p_y - y p_x = m(x\dot{y} - y\dot{x}) = \text{constant.}$$

The entire angular-momentum vector is not generally conserved. Indeed,

$$\vec{\tau} = \vec{r} \times \vec{F} = 2V_0 z (y \hat{x} - x \hat{y}),$$

so τ_x and τ_y need not vanish, whereas $\tau_z = 0$.

4. Additional constants arising from separability

Because the x , y , and z motions are dynamically separable, the energies assigned to the individual Cartesian motions are separately constant:

$$\begin{aligned}E_x &= \frac{p_x^2}{2m} + V_0 x^2 \\E_y &= \frac{p_y^2}{2m} + V_0 y^2 \\E_z &= \frac{p_z^2}{2m}.\end{aligned}$$

For example,

$$\frac{dE_x}{dt} = \frac{p_x \dot{p}_x}{m} + 2V_0 x \dot{x} = \dot{x}(-2V_0 x) + 2V_0 x \dot{x} = 0,$$

and similarly $dE_y/dt = 0$. Also, E_z is constant because p_z is constant. These quantities satisfy

$$E = E_x + E_y + E_z,$$

so they are not all independent of the total energy. The symmetry-generated quantities normally requested are E , p_z , and L_z .

For $V_0 > 0$, defining

$$\omega^2 = \frac{2V_0}{m},$$

the transverse equations become

$$\ddot{x} + \omega^2 x = 0, \quad \ddot{y} + \omega^2 y = 0.$$

Thus the transverse motion is an isotropic harmonic oscillation, while the z -motion is free. This provides a plausibility check: isotropy in the x - y plane requires conservation of L_z , and absence of any z -directed force requires conservation of p_z .

Dimensionally,

$$[V_0] = \frac{[V]}{[x^2]} = \frac{\text{J}}{\text{m}^2} = \text{kg s}^{-2},$$

so $V_0 x^2$ has the dimension of energy and $2V_0 x$ has the dimension of force. Moreover,

$$\left[\frac{2V_0}{m} \right] = \text{s}^{-2},$$

as required for ω^2 . In the limiting case $V_0 \rightarrow 0$, the force vanishes and all three components of linear momentum and all three components of angular momentum become conserved, consistently enlarging the set of conservation laws for a free particle.

CONCLUSION:

For a particle moving in

$$V(\vec{r}) = V_0(x^2 + y^2),$$

the principal conserved physical quantities are

$$E = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + V_0(x^2 + y^2), \quad p_z = m\dot{z}, \quad L_z = x p_y - y p_x.$$

They follow, respectively, from invariance under time translations, translations along the z -axis, and rotations about the z -axis. The direction associated with the conserved linear momentum is the z -direction, and the conserved angular-momentum component is the component along the z -axis. The separate quantities

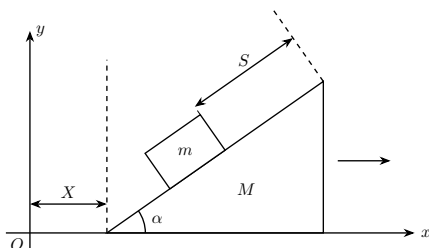
$$E_x = \frac{p_x^2}{2m} + V_0 x^2, \quad E_y = \frac{p_y^2}{2m} + V_0 y^2, \quad E_z = \frac{p_z^2}{2m}$$

are also constants of motion because the Cartesian equations are separable, with $E = E_x + E_y + E_z$. In general, neither the full linear-momentum vector nor the full angular-momentum vector is conserved. For $V_0 > 0$, the particle executes bounded isotropic harmonic motion transverse to the z -axis with $\omega = \sqrt{2V_0/m}$, together with uniform motion parallel to the z -axis;

for $V_0 < 0$, the transverse motion is unstable, but the stated conservation laws remain valid.

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Q8. A small block of mass m slides without friction down a wedge-shaped block of mass M and opening angle α as depicted below.



The wedge-shaped block itself slides along the horizontal frictionless floor along the +ve x -direction. Find the horizontal acceleration $\ddot{X} = d^2X/dt^2$ of the wedge-shaped block by the Lagrangian method and using the displacements X and S of the blocks as generalized coordinates.

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INTRODUCTION:

The required quantity is the horizontal acceleration of the wedge in the inertial frame fixed to the floor. Let the positive x -axis point horizontally to the right and the positive y -axis vertically upward. The coordinate X denotes the horizontal displacement of a reference point fixed to the wedge. The angle α is the angle made by the inclined face with the horizontal.

Let S be the displacement of the small block measured along the inclined face from a fixed upper point of the wedge, with S increasing as the block slides downward and to the left. Thus, increasing S has horizontal and vertical components $-S \cos \alpha$ and $-S \sin \alpha$, respectively.

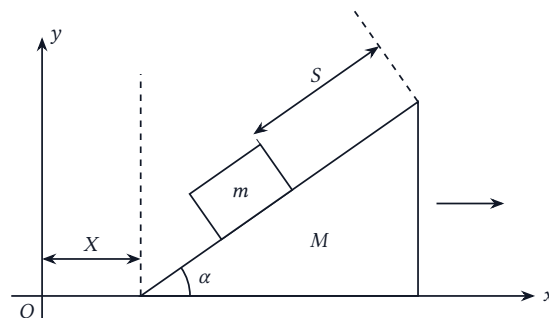
The floor and the contact between the two blocks are frictionless, both blocks are treated as rigid, and the small block remains in contact with the incline.

Hence the only generalized coordinates are X and S . The governing principle is Lagrange's equation,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \quad L = T - V.$$

SOLUTION:

The configuration and the coordinate conventions are shown below.



If (x_m, y_m) are the coordinates of the small block, then, apart from constants fixed by the choice of reference point,

$$x_m = X - S \cos \alpha + \text{constant}$$

$$y_m = -S \sin \alpha + \text{constant}.$$

Therefore,

$$\dot{x}_m = \dot{X} - \dot{S} \cos \alpha, \quad \dot{y}_m = -\dot{S} \sin \alpha.$$

The kinetic energy of the wedge is

$$T_M = \frac{1}{2} M \dot{X}^2,$$

while that of the small block is

$$T_m = \frac{1}{2} m (\dot{x}_m^2 + \dot{y}_m^2)$$

$$= \frac{1}{2} m [(\dot{X} - \dot{S} \cos \alpha)^2 + \dot{S}^2 \sin^2 \alpha]$$

$$= \frac{1}{2} m \dot{X}^2 - m \dot{X} \dot{S} \cos \alpha + \frac{1}{2} m \dot{S}^2.$$

Hence the total kinetic energy is

$$T = \frac{1}{2} (M + m) \dot{X}^2 - m \dot{X} \dot{S} \cos \alpha + \frac{1}{2} m \dot{S}^2.$$

Taking the gravitational potential energy to vanish at any convenient reference level,

$$V = mgy_m = -mgS \sin \alpha + \text{constant}.$$

The irrelevant additive constant may be omitted, giving the Lagrangian

$$L = \frac{1}{2} (M + m) \dot{X}^2 - m \dot{X} \dot{S} \cos \alpha + \frac{1}{2} m \dot{S}^2 + mgS \sin \alpha.$$

Since X does not occur explicitly in L , it is a cyclic coordinate. Its Lagrange equation gives

$$\frac{d}{dt} [(M + m) \dot{X} - m \dot{S} \cos \alpha] = 0,$$

or

$$(M + m)\ddot{X} - m\ddot{S} \cos \alpha = 0. \quad (1)$$

This equation is also the statement of conservation of the total horizontal momentum, as expected because there is no external horizontal force.

For the coordinate S ,

$$\frac{\partial L}{\partial \dot{S}} = m\dot{S} - m\dot{X} \cos \alpha, \quad \frac{\partial L}{\partial S} = mg \sin \alpha.$$

Thus,

$$m\ddot{S} - m\ddot{X} \cos \alpha - mg \sin \alpha = 0,$$

and hence

$$\ddot{S} = g \sin \alpha + \ddot{X} \cos \alpha. \quad (2)$$

Substituting equation (2) into equation (1),

$$(M + m)\ddot{X} = m \cos \alpha (g \sin \alpha + \ddot{X} \cos \alpha),$$

$$(M + m - m \cos^2 \alpha) \ddot{X} = mg \sin \alpha \cos \alpha.$$

Using $1 - \cos^2 \alpha = \sin^2 \alpha$,

$$M + m - m \cos^2 \alpha = M + m \sin^2 \alpha.$$

Therefore,

$$\ddot{X} = \frac{mg \sin \alpha \cos \alpha}{M + m \sin^2 \alpha}.$$

For $0 < \alpha < \pi/2$, both $\sin \alpha$ and $\cos \alpha$ are positive, so $\ddot{X} > 0$. The wedge consequently accelerates toward the right, opposite to the leftward horizontal component of the small block's motion. Also, $\ddot{X} \rightarrow 0$ for either $\alpha \rightarrow 0$ or $\alpha \rightarrow \pi/2$, as expected: in the first case gravity has no component along the surface, while in the second case the block's motion has no horizontal component.

CONCLUSION:

The horizontal acceleration of the wedge-shaped block is

$$\ddot{X} = \frac{mg \sin \alpha \cos \alpha}{M + m \sin^2 \alpha}$$

in units of m s^{-2} . For the configuration shown, with $0 < \alpha < \pi/2$ and S increasing downward along the incline, this acceleration is along the positive x -direction, i.e. horizontally to the right. The result applies while both contacts remain frictionless and the small block remains in contact with the inclined face.

CSE • 2021 • 15 MARKS

Q9. A body of mass m at rest splits into two masses m_1 and m_2 by an explosion. After the

split the bodies move with a total kinetic energy T in opposite direction. Show that their relative speed is $\sqrt{\frac{2Tm}{m_1 m_2}}$.

INTRODUCTION:

The demand is to determine the relative speed of the two fragments using conservation of linear momentum and the given total kinetic energy. We work in the initial rest frame of the body and assume non-relativistic motion, negligible external impulse during the explosion, and conservation of mass in the form

$$m = m_1 + m_2.$$

Choose the x -axis along the motion of mass m_1 . After the explosion, let

$$\vec{v}_1 = v_1 \hat{x}, \quad \vec{v}_2 = -v_2 \hat{x},$$

where $v_1, v_2 > 0$. Thus, the fragments move in opposite directions. Their relative velocity is

$$\vec{v}_{12} = \vec{v}_1 - \vec{v}_2 = (v_1 + v_2) \hat{x},$$

so their relative speed is $v_{\text{rel}} = v_1 + v_2$.

SOLUTION:

Since the original body is at rest, its initial momentum is zero. The explosion is an internal process and the external impulse during it is negligible; hence linear momentum is conserved:

$$m_1 \vec{v}_1 + m_2 \vec{v}_2 = 0.$$

Using the stated sign convention,

$$m_1 v_1 - m_2 v_2 = 0,$$

and therefore

$$m_1 v_1 = m_2 v_2.$$

Let the common magnitude of the fragment momenta be p . Then

$$p = m_1 v_1 = m_2 v_2, \quad v_1 = \frac{p}{m_1}, \quad v_2 = \frac{p}{m_2}.$$

The total kinetic energy after the explosion is

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2.$$

Substituting the velocities in terms of p ,

$$T = \frac{1}{2} m_1 \left(\frac{p}{m_1} \right)^2 + \frac{1}{2} m_2 \left(\frac{p}{m_2} \right)^2,$$

so

$$T = \frac{p^2}{2} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) = \frac{p^2}{2} \frac{m_1 + m_2}{m_1 m_2}.$$

Since $m_1 + m_2 = m$,

$$T = \frac{p^2 m}{2 m_1 m_2},$$

which gives

$$p^2 = \frac{2Tm_1m_2}{m}.$$

Because the velocities are oppositely directed, the relative speed is the sum of their individual speeds:

$$v_{\text{rel}} = v_1 + v_2.$$

Hence

$$v_{\text{rel}} = \frac{p}{m_1} + \frac{p}{m_2} = p \left(\frac{1}{m_1} + \frac{1}{m_2} \right) = p \frac{m}{m_1m_2}.$$

Squaring and substituting for p^2 ,

$$v_{\text{rel}}^2 = p^2 \frac{m^2}{m_1^2m_2^2} = \left(\frac{2Tm_1m_2}{m} \right) \frac{m^2}{m_1^2m_2^2} = \frac{2Tm}{m_1m_2}.$$

Therefore,

$$v_{\text{rel}} = \sqrt{\frac{2Tm}{m_1m_2}}.$$

Equivalently, introducing the reduced mass

$$\mu = \frac{m_1m_2}{m_1 + m_2} = \frac{m_1m_2}{m},$$

the total kinetic energy in the zero-momentum frame is

$$T = \frac{1}{2} \mu v_{\text{rel}}^2,$$

which immediately gives the same result.

The dimensional check is

$$\left[\frac{Tm}{m_1m_2} \right] = \frac{(ML^2T^{-2})M}{M^2} = L^2T^{-2},$$

so its square root has the dimension of speed. Also, as $T \rightarrow 0$, the relative speed tends to zero, as physically required. For $m_1 = m_2 = m/2$, momentum conservation gives equal and opposite speeds, and the formula reduces consistently to

$$v_{\text{rel}} = \sqrt{\frac{8T}{m}}.$$

CONCLUSION:

The fragments move in opposite directions along their common line of motion, with momenta equal in magnitude and opposite in direction. Under the assumptions of negligible external impulse, non-relativistic motion, and $m = m_1 + m_2$, their relative speed is

$$v_{\text{rel}} = \sqrt{\frac{2Tm}{m_1m_2}}.$$

The relative velocity of m_1 with respect to m_2 is directed along the motion of m_1 , while that of m_2 with

respect to m_1 has the same magnitude and the opposite direction.

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Q10. During take-off, an aircraft accelerates horizontally in a straight line at a rate A . A small bob of mass m is suspended on a string attached to the roof of the cabin, and a hydrogen balloon (total mass m) is tethered to the floor by a string. For each, determine the tension in the string and the equilibrium angle θ between string and vertical. Draw a neat diagram to explain your answer.

INTRODUCTION:

Take the aircraft acceleration to be constant and directed along the positive horizontal x -axis:

$$\mathbf{A} = A\hat{\mathbf{x}},$$

with the positive y -axis vertically upward. The calculation will be made in the non-inertial frame fixed to the aircraft. In this frame, every object of mass M experiences the pseudo-force

$$\mathbf{F}_p = -MA\hat{\mathbf{x}}.$$

Consequently, gravity and the pseudo-force together correspond to the effective gravitational acceleration

$$\mathbf{g}_{\text{eff}} = -A\hat{\mathbf{x}} - g\hat{\mathbf{y}}, \quad g_{\text{eff}} = \sqrt{g^2 + A^2}.$$

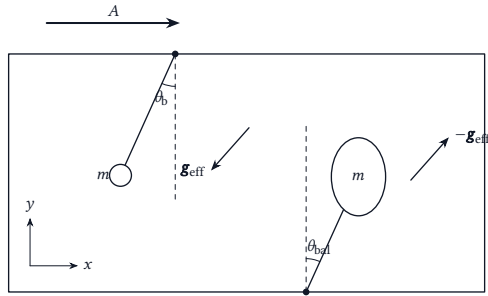
The strings are assumed light, inextensible, and taut. The bob and the balloon are in equilibrium relative to the aircraft. The bob is treated as a point mass. For the balloon, let V be its displaced volume and let ρ be the local density of the cabin air, taken approximately uniform over the dimensions of the balloon. The total balloon mass m includes the hydrogen, envelope, and all attached parts. The cabin air is assumed to have reached mechanical equilibrium in the accelerating frame, so that its pressure gradient is determined by gravity and the horizontal pseudo-force.

For the bob, θ_b is measured from the vertically downward line through the roof attachment. For the balloon, θ_{bal} is measured from the vertically upward line through the floor attachment. Angles toward the positive x -direction are described as forward, while angles toward the negative x -direction are described as backward.

SOLUTION:

1. The geometry and sign convention are shown below. The aircraft accelerates to the right.

The bob is displaced backward, whereas a sufficiently buoyant balloon is displaced forward.



The arrow A points to the right and denotes the actual acceleration of the aircraft. The effective gravity $\mathbf{g}_{\text{eff}}$ points downward and backward. The pressure-produced buoyant force on the balloon will be shown to point opposite to $\mathbf{g}_{\text{eff}}$, namely upward and forward.

2. For the bob, the real forces are its weight

$$\mathbf{W} = -mg\hat{\mathbf{y}}$$

and the tension exerted by the roof string. Since the bob lies behind the roof attachment, the tension on the bob points upward and forward:

$$\mathbf{T}_b = T_b \sin \theta_b \hat{\mathbf{x}} + T_b \cos \theta_b \hat{\mathbf{y}}.$$

The pseudo-force in the aircraft frame is

$$\mathbf{F}_p = -mA\hat{\mathbf{x}}.$$

Equilibrium relative to the aircraft requires

$$\mathbf{T}_b + \mathbf{W} + \mathbf{F}_p = \mathbf{0}.$$

Thus

$$T_b \sin \theta_b \hat{\mathbf{x}} + T_b \cos \theta_b \hat{\mathbf{y}} - mA\hat{\mathbf{x}} - mg\hat{\mathbf{y}} = \mathbf{0}.$$

Equating the horizontal and vertical components gives

$$T_b \sin \theta_b = mA, \quad T_b \cos \theta_b = mg.$$

Dividing the first equation by the second,

$$\tan \theta_b = \frac{A}{g},$$

and hence

$$\theta_b = \tan^{-1}\left(\frac{A}{g}\right).$$

Squaring and adding the component equations,

$$T_b^2 (\sin^2 \theta_b + \cos^2 \theta_b) = m^2 A^2 + m^2 g^2,$$

so that

$$T_b = m\sqrt{g^2 + A^2}.$$

Equivalently, the string aligns the bob with the effective gravity:

$$\mathbf{g}_{\text{eff}} = -A\hat{\mathbf{x}} - g\hat{\mathbf{y}}.$$

Its horizontal component is backward, confirming that the bob must trail behind the accelerating aircraft.

3. For the balloon, ordinary vertical buoyancy alone is insufficient because the accelerating cabin air also has a horizontal pressure gradient. To derive the pressure gradient, consider a small cabin-air element of density ρ at rest in the aircraft frame. The forces per unit volume are the pressure force

$$-\nabla p,$$

gravity

$$-\rho g\hat{\mathbf{y}},$$

and the pseudo-force

$$-\rho A\hat{\mathbf{x}}.$$

Static equilibrium of the air therefore gives

$$-\nabla p - \rho A\hat{\mathbf{x}} - \rho g\hat{\mathbf{y}} = \mathbf{0},$$

or

$$\boxed{\nabla p = -\rho A\hat{\mathbf{x}} - \rho g\hat{\mathbf{y}} = \rho \mathbf{g}_{\text{eff}}}.$$

Thus pressure decreases both upward and forward.

The net pressure force on a body occupying volume V is

$$\mathbf{F}_{\text{pressure}} = -\oint_{\partial V} p \mathbf{n} dS,$$

where $\mathbf{n}$ is the outward unit normal. Applying the divergence theorem component by component, and taking the pressure gradient as approximately constant over the balloon,

$$\mathbf{F}_{\text{pressure}} = -\int_V \nabla p dV = -V\nabla p.$$

Using the pressure-gradient equation,

$$\mathbf{F}_{\text{pressure}} = \rho V A\hat{\mathbf{x}} + \rho V g\hat{\mathbf{y}}.$$

This is the generalized buoyant force in the accelerating cabin. It is directed upward and forward, opposite to $\mathbf{g}_{\text{eff}}$, and has magnitude

$$F_{\text{pressure}} = \rho V \sqrt{g^2 + A^2}.$$

The weight and pseudo-force acting on the total balloon mass m are

$$\mathbf{W} + \mathbf{F}_p = -mg\hat{\mathbf{y}} - mA\hat{\mathbf{x}}.$$

Therefore, before including the tether tension, the resultant force on the balloon is

$$\begin{aligned}\mathbf{F}_{\text{free}} &= \mathbf{F}_{\text{pressure}} + \mathbf{W} + \mathbf{F}_p \\ &= (\rho V - m)A\hat{\mathbf{x}} + (\rho V - m)g\hat{\mathbf{y}} \\ &= (\rho V - m)(A\hat{\mathbf{x}} + g\hat{\mathbf{y}}).\end{aligned}$$

For a balloon that can be held by a taut floor tether,

$$\rho V > m.$$

Under this condition, $\mathbf{F}_{\text{free}}$ points upward and forward. The balloon consequently displaces forward until the floor tether pulls it downward and backward.

The tension exerted on the balloon is directed from the balloon toward the floor attachment. Its components are

$$\mathbf{T}_{\text{bal}} = -T_{\text{bal}} \sin \theta_{\text{bal}} \hat{\mathbf{x}} - T_{\text{bal}} \cos \theta_{\text{bal}} \hat{\mathbf{y}}.$$

Equilibrium requires

$$\mathbf{T}_{\text{bal}} + \mathbf{F}_{\text{free}} = \mathbf{0},$$

so that

$$T_{\text{bal}} \sin \theta_{\text{bal}} = (\rho V - m)A,$$

and

$$T_{\text{bal}} \cos \theta_{\text{bal}} = (\rho V - m)g.$$

Dividing these equations gives

$$\tan \theta_{\text{bal}} = \frac{A}{g},$$

hence

$$\theta_{\text{bal}} = \tan^{-1}\left(\frac{A}{g}\right).$$

Squaring and adding gives

$$T_{\text{bal}}^2 = (\rho V - m)^2(A^2 + g^2),$$

and, since $\rho V - m > 0$,

$$T_{\text{bal}} = (\rho V - m)\sqrt{g^2 + A^2}.$$

If

$$B_0 = \rho V g$$

is the ordinary buoyant force in a non-accelerating cabin, then

$$\rho V - m = \frac{B_0 - mg}{g}.$$

The balloon tension can consequently also be written as

$$T_{\text{bal}} = (B_0 - mg)\sqrt{1 + \frac{A^2}{g^2}}.$$

If $T_0 = B_0 - mg$ is its floor-tether tension when $A = 0$, then

$$T_{\text{bal}} = T_0 \sqrt{1 + \frac{A^2}{g^2}}.$$

Thus the total balloon mass m by itself does not determine its tension. The displaced volume V , the air density ρ , the ordinary buoyant force B_0 , or the stationary-cabin tension T_0 must additionally be known.

4. The limiting cases, dimensions, and physical directions provide checks on the results. When $A \rightarrow 0$,

$$\theta_{\text{b}}, \theta_{\text{bal}} \rightarrow 0,$$

and

$$T_{\text{b}} \rightarrow mg, \quad T_{\text{bal}} \rightarrow (\rho V - m)g = \rho V g - mg.$$

These are precisely the ordinary vertical-equilibrium results.

When $A \gg g$,

$$\tan \theta \simeq \frac{A}{g} \gg 1, \quad \theta \rightarrow \frac{\pi}{2},$$

so both strings become nearly horizontal. The bob lies backward because the effective gravity points backward, while the balloon lies forward because its net buoyancy points opposite to the effective gravity.

The dimensions are

$$\left[m\sqrt{g^2 + A^2}\right] = [\text{mass}][\text{acceleration}] = [\text{force}],$$

and similarly

$$\left[(\rho V - m)\sqrt{g^2 + A^2}\right] = [\text{force}].$$

Thus both tensions have the required SI unit of newton.

If $\rho V = m$, the pressure force exactly balances the balloon's weight and pseudo-force:

$$\mathbf{F}_{\text{free}} = \mathbf{0}, \quad T_{\text{bal}} = 0.$$

The tether is then slack and its equilibrium angle is not determined. If $\rho V < m$, the free resultant is downward and backward. A floor tether, which can only pull the balloon toward the floor, cannot balance that resultant with a taut string in the stated configuration.

CONCLUSION:

For the bob suspended from the roof,

$$T_b = m\sqrt{g^2 + A^2} \text{ N}, \quad \theta_b = \tan^{-1}\left(\frac{A}{g}\right).$$

The angle is dimensionless and may be expressed in radians or degrees. The bob and its string slope backward, opposite to the direction of aircraft acceleration.

For a balloon of total mass m and displaced volume V , in cabin air of density ρ , a taut floor-tether equilibrium requires

$$\rho V > m.$$

Under this condition,

$$T_{\text{bal}} = (\rho V - m)\sqrt{g^2 + A^2} \text{ N}, \quad \theta_{\text{bal}} = \tan^{-1}\left(\frac{A}{g}\right).$$

Equivalently,

$$T_{\text{bal}} = (B_0 - mg)\sqrt{1 + \frac{A^2}{g^2}} \text{ N}, \quad B_0 = \rho V g.$$

The balloon and its string slope forward, in the direction of aircraft acceleration. Thus the bob and the buoyant balloon make equal angular magnitudes with their respective vertical reference lines, but they lean in opposite horizontal directions. The balloon tension is not determined by the stated mass m alone; its displaced-air mass ρV , ordinary buoyant force B_0 , or stationary-cabin tether tension must also be specified.



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