



UPSC PHYSICS OPTIONAL for CSE & IFoS

ABHI PHYSICS

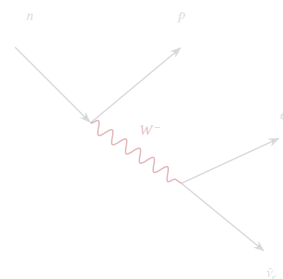
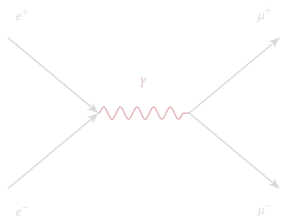
$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$E_n = -\frac{13.6}{n^2} \text{ eV}$$

PREVIOUS YEAR QUESTIONS



41 YEARS QUESTION BANK | 1986-2026

CSE + IFoS

SUBJECT

Quantum Mechanics

10 SOLVED QUESTIONS • CSE + IFoS

$$n \rightarrow p + e^- + \bar{\nu}_e$$

$$E^2 = p^2 c^2 + m^2 c^4$$

$$\partial_\mu j^\mu = 0$$

SUBJECT-WISE • TOPIC-WISE • SOURCE-TAGGED

For Civil Services and Indian Forest Service Examinations

Sample Preview extbar 10 Solved Questions

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CSE • 2026 • 5 MARKS

Q1. The de Broglie wavelength of a particle in thermal equilibrium at a temperature of 27°C is λ . At what temperature will it be $\frac{\lambda}{2}$?

INTRODUCTION:

We have to determine the temperature at which the thermal de Broglie wavelength of the same non-relativistic particle becomes half its initial value. For a particle of mass m ,

$$\lambda_{\text{dB}} = \frac{h}{p},$$

where h is Planck's constant and p is the magnitude of its momentum. In thermal equilibrium, any fixed conventional measure of the characteristic thermal momentum, such as the root-mean-square momentum, is proportional to $\sqrt{T}$, where T is the absolute temperature. Hence the thermal de Broglie wavelength varies as $T^{-1/2}$.

Let

$$T_1 = 27^\circ\text{C} = 300\text{ K}, \quad \lambda_1 = \lambda,$$

and let T_2 be the temperature at which

$$\lambda_2 = \frac{\lambda}{2}.$$

SOLUTION:

For a non-relativistic particle in thermal equilibrium, the equipartition theorem gives

$$\frac{p_{\text{rms}}^2}{2m} = \frac{3}{2}k_{\text{B}}T,$$

so that

$$p_{\text{rms}} = \sqrt{3mk_{\text{B}}T}.$$

Therefore,

$$\lambda_{\text{dB}} = \frac{h}{p_{\text{rms}}} = \frac{h}{\sqrt{3mk_{\text{B}}T}} \propto \frac{1}{\sqrt{T}}.$$

For the same particle at the two temperatures,

$$\frac{\lambda_2}{\lambda_1} = \sqrt{\frac{T_1}{T_2}}.$$

Substituting $\lambda_2 = \lambda/2$, $\lambda_1 = \lambda$, and $T_1 = 300\text{ K}$,

$$\frac{1}{2} = \sqrt{\frac{300}{T_2}}.$$

Squaring both sides,

$$\frac{1}{4} = \frac{300}{T_2},$$

and hence

$$T_2 = 4(300\text{ K}) = 1200\text{ K}.$$

Expressed on the Celsius scale,

$$t_2 = T_2 - 273 \approx 1200 - 273 = 927^\circ\text{C}.$$

Thus halving the wavelength requires the characteristic momentum to double; since $p_{\text{rms}} \propto \sqrt{T}$, the absolute temperature must increase by a factor of four.

CONCLUSION:

The required temperature is

$$T_2 = 1200\text{ K} \approx 927^\circ\text{C}.$$

This result assumes the same particle remains in the non-relativistic thermal regime.

CSE • 2026 • 10 MARKS

Q2. If the position and momentum operators in 1-D are represented by $\hat{x}$ and $\hat{p}$ respectively, calculate the commutator $[\hat{x}^2, \cos(\hat{p})]$.

INTRODUCTION:

The required commutator is evaluated using the canonical position-momentum relation

$$[\hat{x}, \hat{p}] = i\hbar.$$

For any sufficiently differentiable function $f(\hat{p})$, the corresponding operator identity is

$$[\hat{x}, f(\hat{p})] = i\hbar f'(\hat{p}).$$

Here $\cos(\hat{p})$ is understood as an operator function defined by its power series or, equivalently, by the spectral calculus. Its argument is assumed dimensionless; if a momentum scale p_0 is required, $\hat{p}$ is to be replaced by $\hat{p}/p_0$.

SOLUTION:

Using the product rule for commutators,

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B},$$

we obtain

$$[\hat{x}^2, \cos(\hat{p})] = \hat{x}[\hat{x}, \cos(\hat{p})] + [\hat{x}, \cos(\hat{p})]\hat{x}.$$

Since

$$[\hat{x}, \cos(\hat{p})] = i\hbar \frac{d}{d\hat{p}} \cos(\hat{p}) = -i\hbar \sin(\hat{p}),$$

it follows that

$$[\hat{x}^2, \cos(\hat{p})] = -i\hbar [\hat{x} \sin(\hat{p}) + \sin(\hat{p})\hat{x}].$$

Thus the most symmetric form of the answer is

$$[\hat{x}^2, \cos(\hat{p})] = -i\hbar\{\hat{x}, \sin(\hat{p})\},$$

where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ denotes the anticommutator.

An equivalent normally ordered form can be found from

$$[\hat{x}, \sin(\hat{p})] = i\hbar \cos(\hat{p}),$$

which gives

$$\hat{x} \sin(\hat{p}) = \sin(\hat{p})\hat{x} + i\hbar \cos(\hat{p}).$$

Therefore,

$$[\hat{x}^2, \cos(\hat{p})] = -2i\hbar \sin(\hat{p})\hat{x} + \hbar^2 \cos(\hat{p}).$$

Equivalently, with $\hat{x}$ placed on the left,

$$[\hat{x}^2, \cos(\hat{p})] = -2i\hbar \hat{x} \sin(\hat{p}) - \hbar^2 \cos(\hat{p}).$$

CONCLUSION:

The required commutator is

$$[\hat{x}^2, \cos(\hat{p})] = -i\hbar [\hat{x} \sin(\hat{p}) + \sin(\hat{p})\hat{x}] = -i\hbar\{\hat{x}, \sin(\hat{p})\}.$$

The equivalent ordered expressions are

$$[\hat{x}^2, \cos(\hat{p})] = -2i\hbar \sin(\hat{p})\hat{x} + \hbar^2 \cos(\hat{p})$$

and

$$[\hat{x}^2, \cos(\hat{p})] = -2i\hbar \hat{x} \sin(\hat{p}) - \hbar^2 \cos(\hat{p}).$$

CSE • 2026 • 10 MARKS

Q3. A particle is in the normalized state ψ which is a superposition of the energy eigenstates ψ_1 and ψ_2 with energies 10 eV and 30 eV, respectively. The average value of the energy of the particle in the state ψ is 24 eV. Find the state ψ in terms of ψ_1 and ψ_2 .

INTRODUCTION:

Let the Hamiltonian satisfy

$$\hat{H}\psi_1 = E_1\psi_1, \quad \hat{H}\psi_2 = E_2\psi_2,$$

where

$$E_1 = 10 \text{ eV}, \quad E_2 = 30 \text{ eV}.$$

Since eigenstates belonging to distinct eigenvalues of a Hermitian Hamiltonian are orthogonal, we take

$$\langle \psi_i | \psi_j \rangle = \delta_{ij}.$$

The most general superposition is

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle,$$

where $c_1, c_2 \in \mathbb{C}$. Normalization and the given expectation value of energy will determine the probabilities $|c_1|^2$ and $|c_2|^2$; the relative phase cannot be determined from an energy expectation value alone.

SOLUTION:

Normalization gives

$$\langle \psi | \psi \rangle = |c_1|^2 + |c_2|^2 = 1.$$

The expectation value of the Hamiltonian is

$$\begin{aligned} \langle E \rangle &= \langle \psi | \hat{H} | \psi \rangle \\ &= |c_1|^2 E_1 + |c_2|^2 E_2, \end{aligned}$$

because the cross terms vanish by orthogonality. Hence

$$10|c_1|^2 + 30|c_2|^2 = 24,$$

with all energies measured in eV.

Using $|c_1|^2 = 1 - |c_2|^2$,

$$10(1 - |c_2|^2) + 30|c_2|^2 = 24,$$

so that

$$10 + 20|c_2|^2 = 24$$

and therefore

$$|c_2|^2 = \frac{14}{20} = \frac{7}{10}.$$

Consequently,

$$|c_1|^2 = 1 - \frac{7}{10} = \frac{3}{10}.$$

Thus the coefficient magnitudes are

$$|c_1| = \sqrt{\frac{3}{10}}, \quad |c_2| = \sqrt{\frac{7}{10}}.$$

After removing an physically irrelevant common phase, the remaining relative phase may be denoted by ϕ , giving

$$|\psi\rangle = \sqrt{\frac{3}{10}} |\psi_1\rangle + e^{i\phi} \sqrt{\frac{7}{10}} |\psi_2\rangle, \quad \phi \in \mathbb{R}.$$

Indeed, this state yields

$$\langle E \rangle = \frac{3}{10}(10 \text{ eV}) + \frac{7}{10}(30 \text{ eV}) = 3 \text{ eV} + 21 \text{ eV} = 24 \text{ eV},$$

as required.

CONCLUSION:

Up to an arbitrary overall phase, the normalized state is

$$\psi = \sqrt{\frac{3}{10}} \psi_1 + e^{i\phi} \sqrt{\frac{7}{10}} \psi_2, \quad \phi \in \mathbb{R}.$$

Therefore, an energy measurement gives 10 eV with probability 3/10 and 30 eV with probability 7/10. The specified mean energy fixes these probabilities but does not fix the relative phase ϕ .

CSE • 2025 • 10 MARKS

Q4. Explain how the uncertainty in position is different from the uncertainty or inaccuracy of the measuring instruments.

INTRODUCTION:

The question asks for the distinction between the quantum-mechanical uncertainty in the position of a particle and the uncertainty introduced by an imperfect measuring instrument. Consider an ensemble of identically prepared particles in a fixed reference frame. Let the position observable be represented by the operator $\hat{x}$, and let the normalized state of each particle be $|\psi\rangle$. The intrinsic position uncertainty is defined statistically by

$$\Delta x = \sqrt{\langle (\hat{x} - \langle \hat{x} \rangle)^2 \rangle} = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2},$$

where $\langle \hat{x} \rangle = \langle \psi | \hat{x} | \psi \rangle$. Instrumental inaccuracy, on the other hand, is a property of the measurement apparatus and its calibration, resolution, stability, and interaction with the system.

SOLUTION:

In classical physics, a particle is ordinarily assumed to possess a definite position at every instant. If repeated measurements give different values, the spread is attributed to imperfect instruments, environmental disturbances, or lack of experimental control. In principle, one expects this spread to be reduced indefinitely by improving the apparatus.

Quantum-mechanical position uncertainty has a different origin. If the particle is described in the position representation by a normalized wavefunction $\psi(x)$, then

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1,$$

and the probability of finding it between x and $x + dx$ is

$$dP = |\psi(x)|^2 dx.$$

The expectation values are

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\psi(x)|^2 dx, \quad \langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx,$$

so that

$$\Delta x = \left[\int_{-\infty}^{\infty} (x - \langle x \rangle)^2 |\psi(x)|^2 dx \right]^{1/2}.$$

Thus, Δx measures the spread of the position probabilities inherent in the prepared quantum state. Even an ideal apparatus, assumed to have perfect resolution and calibration, generally produces a distribution of position outcomes when the experiment is repeated on identically prepared systems. This spread is not merely the observer's ignorance of a pre-existing exact value in the classical sense; it is part of the quantum description of the state.

The intrinsic character of this uncertainty is expressed by the position-momentum uncertainty relation. Since

$$[\hat{x}, \hat{p}_x] = i\hbar,$$

the Robertson relation gives

$$\Delta x \Delta p_x \geq \frac{1}{2} |\langle [\hat{x}, \hat{p}_x] \rangle| = \frac{\hbar}{2}.$$

Consequently, preparing a state with a smaller Δx necessarily requires a larger spread Δp_x . This limitation follows from the non-commuting structure of the observables and remains even for an ideal measuring instrument. It is therefore not a statement about defective experimental technique.

Instrumental uncertainty may instead be represented by a measurement error ϵ_x , determined, for example, by the detector's finite scale division, point-spread width, noise, parallax, calibration error, or systematic offset. It answers the question: how faithfully does the apparatus report the position outcome? Such errors may be classified as follows:

1. Random instrumental errors cause repeated readings to fluctuate and may often be reduced by better resolution, noise control, or statistical averaging.
2. Systematic errors, such as a zero error or incorrect calibration, shift the readings in a definite manner and can often be estimated and corrected by calibration.
3. Measurement disturbance arises when the measuring interaction changes the state of the particle. It is apparatus-dependent and should not be identified automatically with the intrinsic standard deviation Δx .

For an unbiased apparatus whose random error is statistically independent of the intrinsic position distribution, a simple model is

$$x_{\text{read}} = x + \delta x_{\text{inst}}, \quad \langle \delta x_{\text{inst}} \rangle = 0.$$

The variance of the recorded readings is then

$$\sigma_{\text{read}}^2 = (\Delta x)^2 + \epsilon_x^2,$$

where

$$\epsilon_x^2 = \langle (\delta x_{\text{inst}})^2 \rangle.$$

This relation shows that the experimentally observed spread may contain both intrinsic quantum uncertainty and instrumental imprecision. If the apparatus is progressively improved so that

$$\epsilon_x \longrightarrow 0,$$

then

$$\sigma_{\text{read}} \longrightarrow \Delta x,$$

not necessarily to zero. Hence the residual spread obtained with an ideal instrument is the position uncertainty of the quantum state.

Conversely, Δx can vanish only for an idealized exact position eigenstate. Such a state is non-normalizable over the whole real line and, consistently with $\Delta x \Delta p_x \geq \hbar/2$, would require an unbounded momentum spread. Any physically realizable localized wave packet therefore has a finite spatial spread.

The two concepts must also be distinguished operationally. The intrinsic Δx is inferred from the statistics of many measurements on an ensemble of systems prepared in the same state; it is not the uncertainty of a single scale reading. Instrumental inaccuracy characterizes the reliability of the apparatus and can be assessed through calibration and repeated measurements of known standards. Improving the instrument reduces ϵ_x , whereas changing the quantum preparation is required to change Δx .

CONCLUSION:

The uncertainty in position,

$$\Delta x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2},$$

is the intrinsic statistical spread of position outcomes for identically prepared quantum systems. It is constrained by

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}$$

and cannot, in general, be removed by constructing a more accurate apparatus.

Instrumental uncertainty ϵ_x is instead caused by finite resolution, noise, calibration errors, or other limitations of the measuring procedure. It is apparatus-dependent and can in principle be reduced or corrected. For independent unbiased instrumental error,

$$\sigma_{\text{read}}^2 = (\Delta x)^2 + \epsilon_x^2,$$

so an ideal instrument removes the instrumental contribution but still reveals the intrinsic quantum position distribution.

CSE • 2025 • 5 MARKS

Q5. Is it possible for a photon to transfer all its energy to a free electron? Give reasons.

INTRODUCTION:

The question asks whether an isolated free electron can completely absorb a photon,

$$\gamma + e^- \longrightarrow e^-,$$

while conserving both energy and momentum. Assume that the electron is initially at rest in the laboratory frame, with rest mass m , and that the incident photon has energy $E_\gamma = h\nu > 0$ and momentum magnitude $p_\gamma = E_\gamma/c$. After absorption, let the electron have momentum p and total relativistic energy E_e . No other particle or external body is available to carry energy or momentum.

SOLUTION:

If the photon were absorbed completely, conservation of momentum would require

$$p = p_\gamma = \frac{E_\gamma}{c}.$$

Conservation of energy would simultaneously require

$$mc^2 + E_\gamma = E_e.$$

For a free relativistic electron,

$$E_e^2 = p^2 c^2 + m^2 c^4.$$

Using the momentum-conservation result,

$$E_e^2 = \left(\frac{E_\gamma}{c}\right)^2 c^2 + m^2 c^4 = E_\gamma^2 + m^2 c^4.$$

On the other hand, energy conservation gives

$$E_e^2 = (mc^2 + E_\gamma)^2 = m^2 c^4 + E_\gamma^2 + 2mc^2 E_\gamma.$$

Equating these two expressions for E_e^2 yields

$$m^2 c^4 + E_\gamma^2 = m^2 c^4 + E_\gamma^2 + 2mc^2 E_\gamma,$$

and hence

$$2mc^2 E_\gamma = 0.$$

Since $m \neq 0$, this condition can be satisfied only for

$$E_\gamma = 0.$$

Thus, for any real photon with $E_\gamma > 0$, simultaneous conservation of energy and momentum is impossible.

The result is independent of the frame chosen. If the initial electron has four-momentum P^μ , the photon has four-momentum K^μ , and the final electron has four-momentum $P'^\mu = P^\mu + K^\mu$, then

$$P^2 = P'^2 = m^2 c^2, \quad K^2 = 0.$$

Therefore,

$$P'^2 = (P + K)^2 = P^2 + 2P \cdot K + K^2 = m^2 c^2 + 2P \cdot K.$$

For a future-directed massive electron and a real photon of positive energy, $P \cdot K > 0$; hence $P'^2 > m^2 c^2$, contradicting the required mass shell of a free electron.

Complete photon absorption can occur only if another object participates—for example, a bound atom, a nucleus, or a solid lattice—so that the additional system can take up the recoil momentum while conserving total energy.

CONCLUSION:

No non-zero-energy photon can transfer all its energy to a single isolated free electron. Complete absorption would violate simultaneous conservation of relativistic energy and momentum. A recoil partner or an external field is necessary to absorb the excess momentum; otherwise the photon can transfer only part of its energy, as in Compton scattering, and a scattered photon must remain.

IFoS • 2025 • 6+2=8 MARKS

Q6. A bullet of mass 0.025 kg is moving with a velocity of 600 m/s . The speed is measured with an accuracy of 0.02% . Find out the uncertainty in x . Also, comment on the result.

INTRODUCTION:

The speed accuracy determines the uncertainty in the bullet's momentum, which in turn limits the precision with which its position can be specified. We use the Heisenberg uncertainty relation

$$\Delta x \Delta p_x \geq \frac{\hbar}{2} = \frac{h}{4\pi}.$$

Here Δx and Δp_x are the position and momentum uncertainties along the direction of motion. The quoted percentage accuracy is interpreted as the uncertainty in the measured speed. Since the bullet's mass is treated as exactly known and its motion is non-relativistic,

$$p_x = mv, \quad \Delta p_x = m \Delta v.$$

The minimum possible position uncertainty corresponds to equality in the uncertainty relation.

SOLUTION:

The fractional uncertainty in speed is

$$\frac{\Delta v}{v} = \frac{0.02}{100} = 2.0 \times 10^{-4}.$$

Hence

$$\Delta v = v(2.0 \times 10^{-4}) = (600 \text{ m s}^{-1})(2.0 \times 10^{-4}) = 0.12 \text{ m s}^{-1}.$$

Therefore, the momentum uncertainty is

$$\begin{aligned} \Delta p_x &= m \Delta v \\ &= (0.025 \text{ kg})(0.12 \text{ m s}^{-1}) \\ &= 3.0 \times 10^{-3} \text{ kg m s}^{-1}. \end{aligned}$$

Using

$$\Delta x_{\min} = \frac{\hbar}{2\Delta p_x},$$

with

$$\hbar = 1.055 \times 10^{-34} \text{ J s},$$

we obtain

$$\begin{aligned} \Delta x_{\min} &= \frac{1.055 \times 10^{-34} \text{ J s}}{2(3.0 \times 10^{-3} \text{ kg m s}^{-1})} \\ &= \frac{1.055 \times 10^{-34} \text{ kg m}^2 \text{ s}^{-1}}{6.0 \times 10^{-3} \text{ kg m s}^{-1}} \\ &= 1.76 \times 10^{-32} \text{ m}. \end{aligned}$$

Equivalently, since the bullet momentum is

$$p = mv = (0.025)(600) = 15 \text{ kg m s}^{-1},$$

its relative momentum uncertainty is

$$\frac{\Delta p_x}{p} = \frac{3.0 \times 10^{-3}}{15} = 2.0 \times 10^{-4},$$

consistent with the stated relative speed uncertainty.

The calculated minimum uncertainty is extraordinarily smaller than the dimensions of a bullet and even much smaller than a nuclear length scale, of order 10^{-15} m . Thus the quantum-mechanical lower bound on its position uncertainty is experimentally and macroscopically negligible. The bullet can therefore be treated as following a well-defined classical trajectory, although it still formally obeys the uncertainty principle.

CONCLUSION:

The speed uncertainty and corresponding momentum uncertainty are

$$\Delta v = 0.12 \text{ m s}^{-1},$$

$$\Delta p_x = 3.0 \times 10^{-3} \text{ kg m s}^{-1}.$$

Consequently, the minimum position uncertainty along the direction of motion is

$$\Delta x_{\min} \approx 1.8 \times 10^{-32} \text{ m}.$$

This is a lower bound obtained by interpreting the quoted speed accuracy as Δv and taking the minimum-uncertainty case. It has no independent spatial direction; it is the scalar spread in position along the chosen direction of motion. Its extremely small magnitude shows why quantum uncertainty is imperceptible for a macroscopic bullet and why classical mechanics provides an excellent description of its motion.

CSE • 2024 • 10 MARKS

Q7. A particle limited to the x -axis has the wave function $\phi(x) = bx^2$ between $x = 0$ and $x = 2$; the wave function $\phi(x) = 0$ elsewhere.

(i) Find the probability that the particle can be found between $x = 1.0$ and $x = 1.5$. (ii) Find the expectation value $\langle x \rangle$ of the particle position.

INTRODUCTION:

The problem requires the probability of finding the particle in a specified interval and the expectation value of its position. In one-dimensional quantum mechanics, the probability density is

$$\rho(x) = |\phi(x)|^2,$$

and a physically admissible wave function must satisfy

$$\int_{-\infty}^{\infty} |\phi(x)|^2 dx = 1.$$

We take the positive x -direction as the coordinate direction and write

$$\phi(x) = \begin{cases} bx^2, & 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

The constant b may be complex, but only $|b|^2$ enters measurable probabilities. An overall phase is physically irrelevant, so b may be chosen real and positive after normalization. If x has dimensions of length, then $[\phi] = L^{-1/2}$ and consequently $[b] = L^{-5/2}$.

SOLUTION:

First normalize the wave function:

$$1 = \int_{-\infty}^{\infty} |\phi(x)|^2 dx = |b|^2 \int_0^2 x^4 dx.$$

Therefore,

$$1 = |b|^2 \left[\frac{x^5}{5} \right]_0^2 = |b|^2 \frac{2^5}{5} = |b|^2 \frac{32}{5},$$

so that

$$|b|^2 = \frac{5}{32}, \quad |b| = \sqrt{\frac{5}{32}}.$$

(i) The probability of finding the particle between $x = 1.0$ and $x = 1.5$ is

$$P(1.0 \leq x \leq 1.5) = \int_1^{1.5} |\phi(x)|^2 dx = |b|^2 \int_1^{1.5} x^4 dx.$$

Substituting the normalization constant,

$$P = \frac{5}{32} \left[\frac{x^5}{5} \right]_1^{1.5} = \frac{1}{32} \left[\left(\frac{3}{2} \right)^5 - 1 \right].$$

Since

$$\left(\frac{3}{2} \right)^5 = \frac{243}{32},$$

we obtain

$$P = \frac{1}{32} \left(\frac{243}{32} - 1 \right) = \frac{1}{32} \left(\frac{211}{32} \right) = \frac{211}{1024}.$$

Numerically,

$$P = \frac{211}{1024} \approx 0.2061,$$

or approximately 20.6%.

(ii) The expectation value of the particle position is

$$\langle x \rangle = \int_{-\infty}^{\infty} \phi^*(x) x \phi(x) dx = \int_0^2 x |\phi(x)|^2 dx.$$

Thus,

$$\langle x \rangle = |b|^2 \int_0^2 x^5 dx = \frac{5}{32} \left[\frac{x^6}{6} \right]_0^2.$$

Hence,

$$\langle x \rangle = \frac{5}{32} \frac{2^6}{6} = \frac{5}{32} \frac{64}{6} = \frac{5}{3}.$$

The result satisfies $0 < \langle x \rangle < 2$, as required because the probability density vanishes outside this interval. Moreover, since $|\phi(x)|^2 \propto x^4$, the probability is weighted toward larger x , so $\langle x \rangle = 5/3 > 1$ is physically reasonable.

CONCLUSION:

The normalized wave function, up to an arbitrary constant phase, is

$$\phi(x) = \begin{cases} \sqrt{\frac{5}{32}} x^2, & 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

The requested results are

$$P(1.0 \leq x \leq 1.5) = \frac{211}{1024} \approx 0.2061$$

and

$$\langle x \rangle = \frac{5}{3} \approx 1.67.$$

Thus, the probability is approximately 20.6%, and the mean position lies toward the upper end of the allowed interval because the probability density increases as x^4 . The value of $\langle x \rangle$ has the same position unit as the stated x -coordinates.

IFoS • 2024 • 8 MARKS

Q8. Determine the size of the hydrogen atom using uncertainty principle. Given that the potential energy of electron $V = \frac{-e^2}{4\pi\epsilon_0 a}$, where a is the distance of the electron from the nucleus.

INTRODUCTION:

The electron cannot remain at rest near the proton: localizing it within a region of characteristic radius a necessarily produces momentum uncertainty and hence kinetic energy. In the proton rest frame, neglecting proton recoil and relativistic effects, the electron has total energy

$$E = K + V, \quad V(a) = -\frac{e^2}{4\pi\epsilon_0 a}.$$

Let m_e be the electron mass. For localization over the distance a , the uncertainty principle gives the order-of-magnitude relation

$$\Delta x \Delta p \sim \hbar, \quad \Delta x \sim a,$$

so that

$$p \sim \Delta p \sim \frac{\hbar}{a}.$$

The uncertainty principle fixes the characteristic atomic length scale; numerical factors of order unity depend on the precise definition of the position spread. The conventional estimate $pa \sim \hbar$ yields the Bohr radius.

SOLUTION:

The kinetic energy associated with the momentum uncertainty is

$$K(a) \sim \frac{p^2}{2m_e} \sim \frac{\hbar^2}{2m_e a^2}.$$

Hence the energy of an electron localized at a characteristic distance a is

$$E(a) \sim \frac{\hbar^2}{2m_e a^2} - \frac{e^2}{4\pi\epsilon_0 a}.$$

The first term increases as a^{-2} when the electron is confined more strongly, whereas the attractive

Coulomb energy decreases as $-a^{-1}$. Their competition therefore permits a stable finite atomic size.

The equilibrium size is found by minimizing $E(a)$:

$$\frac{dE}{da} = -\frac{\hbar^2}{m_e a^3} + \frac{e^2}{4\pi\epsilon_0 a^2} = 0.$$

Thus

$$\frac{\hbar^2}{m_e a^3} = \frac{e^2}{4\pi\epsilon_0 a^2},$$

and therefore

$$a = \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2}.$$

This is the Bohr radius a_0 .

Using

$$\epsilon_0 = 8.854 \times 10^{-12} \text{ F m}^{-1}, \quad \hbar = 1.055 \times 10^{-34} \text{ J s},$$

$$m_e = 9.109 \times 10^{-31} \text{ kg}, \quad e = 1.602 \times 10^{-19} \text{ C},$$

one obtains

$$a_0 = \frac{4\pi(8.854 \times 10^{-12})(1.055 \times 10^{-34})^2}{(9.109 \times 10^{-31})(1.602 \times 10^{-19})^2} \text{ m} \\ \simeq 5.29 \times 10^{-11} \text{ m}.$$

Equivalently,

$$a_0 \simeq 0.529 \text{ Å}.$$

The stationary point is a minimum because

$$\frac{d^2 E}{da^2} = \frac{3\hbar^2}{m_e a^4} - \frac{2e^2}{4\pi\epsilon_0 a^3}.$$

At $a = a_0$, the equilibrium condition gives

$$\frac{e^2}{4\pi\epsilon_0 a_0^3} = \frac{\hbar^2}{m_e a_0^4},$$

and hence

$$\left. \frac{d^2 E}{da^2} \right|_{a_0} = \frac{\hbar^2}{m_e a_0^4} > 0.$$

Thus the uncertainty-induced kinetic energy prevents collapse into the nucleus. Moreover, at the minimum,

$$\frac{e^2}{4\pi\epsilon_0 a_0} = \frac{\hbar^2}{m_e a_0^2} = 2K,$$

so $V = -2K$ and $E = K + V = -K < 0$, consistently describing a bound state.

CONCLUSION:

The uncertainty principle gives the characteristic size of the hydrogen atom as

$$a_0 = \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2} \simeq 5.29 \times 10^{-11} \text{ m} = 0.529 \text{ Å}.$$

This is the Bohr radius, interpreted as the characteristic electron-proton separation in the ground

state. The result follows in the non-relativistic, fixed-nucleus Coulomb approximation, with the uncertainty estimate $pa \sim \hbar$; replacing m_e by the electron-proton reduced mass gives the small finite-nuclear-mass correction.

CSE • 2023 • 15 MARKS

Q9. A particle constrained to move along x -axis in the domain $0 \leq x \leq L$ has a wave function $\psi(x) = \sin\left(\frac{n\pi x}{L}\right)$, where n is an integer. Normalize the wave function and evaluate the expectation value of momentum of the particle.

INTRODUCTION:

The required tasks are to determine the normalization constant and to calculate the expectation value of the momentum operator. Let

$$\psi_N(x) = A \sin\left(\frac{n\pi x}{L}\right), \quad 0 \leq x \leq L,$$

and $\psi_N(x) = 0$ outside this interval. Here $L > 0$, n must be a nonzero integer, and A may be chosen real and positive because an overall constant phase has no physical significance. The normalization condition and the momentum operator in the position representation are

$$\int_0^L |\psi_N(x)|^2 dx = 1, \quad \hat{p}_x = -i\hbar \frac{d}{dx}.$$

The positive x -direction is taken as the positive direction of momentum.

SOLUTION:

Applying the normalization condition,

$$1 = |A|^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx.$$

Using

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2},$$

the integral becomes

$$\begin{aligned} \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx &= \frac{1}{2} \int_0^L \left[1 - \cos\left(\frac{2n\pi x}{L}\right)\right] dx \\ &= \left[\frac{x}{2} - \frac{L}{4n\pi} \sin\left(\frac{2n\pi x}{L}\right)\right]_0^L \\ &= \frac{L}{2}, \end{aligned}$$

since $\sin(2n\pi) = 0$ for integer n . Hence

$$|A|^2 \frac{L}{2} = 1 \quad \Rightarrow \quad |A| = \sqrt{\frac{2}{L}}.$$

Choosing A real and positive gives

$$\psi_N(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right).$$

The expectation value of momentum is

$$\langle p_x \rangle = \int_0^L \psi_N^*(x) \left(-i\hbar \frac{d}{dx}\right) \psi_N(x) dx.$$

Since the normalized wave function is real,

$$\frac{d\psi_N}{dx} = \sqrt{\frac{2}{L}} \frac{n\pi}{L} \cos\left(\frac{n\pi x}{L}\right).$$

Therefore,

$$\begin{aligned} \langle p_x \rangle &= -i\hbar \frac{2n\pi}{L^2} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx \\ &= -i\hbar \frac{2n\pi}{L^2} \left[\frac{L}{2n\pi} \sin^2\left(\frac{n\pi x}{L}\right) \right]_0^L \\ &= -\frac{i\hbar}{L} [\sin^2(n\pi) - \sin^2(0)] \\ &= 0. \end{aligned}$$

Equivalently, for any real wave function vanishing at both boundaries,

$$\langle p_x \rangle = -\frac{i\hbar}{2} [\psi_N^2(x)]_0^L = 0.$$

Although the mean momentum vanishes, the state is not a momentum eigenstate. Indeed,

$$\sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2i} [e^{in\pi x/L} - e^{-in\pi x/L}],$$

so the standing wave contains equal contributions associated with opposite momenta $+n\pi\hbar/L$ and $-n\pi\hbar/L$; their contributions to the mean cancel.

CONCLUSION:

For $L > 0$ and any nonzero integer n , the normalized wave function, up to an arbitrary constant phase, is

$$\psi_N(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad 0 \leq x \leq L$$

and it vanishes outside the allowed interval. Its momentum expectation value is

$$\langle p_x \rangle = 0 \text{ kg m s}^{-1}.$$

Thus the particle has no preferred average direction of motion: the positive- and negative-momentum contributions are balanced. The case $n = 0$ is excluded because it gives the identically zero function, which cannot be normalized.

IFoS • 2023 • 8 MARKS

Q10. Find the condition at which de Broglie wavelength equals the Compton wavelength for a particle.

INTRODUCTION:

For a free particle of rest mass m , the de Broglie wavelength is

$$\lambda_{dB} = \frac{h}{p},$$

where p is the magnitude of its relativistic momentum. Its Compton wavelength is

$$\lambda_C = \frac{h}{mc}.$$

Since the required momentum may be relativistic, we use

$$p = \gamma mv, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}},$$

in the inertial frame in which the particle moves with speed v .

SOLUTION:

Equating the two wavelengths,

$$\lambda_{dB} = \lambda_C,$$

gives

$$\frac{h}{p} = \frac{h}{mc}.$$

Therefore,

$$p = mc.$$

Using $p = \gamma mv$,

$$\gamma mv = mc \implies \gamma v = c.$$

Writing $\beta = v/c$, this condition becomes

$$\frac{\beta}{\sqrt{1 - \beta^2}} = 1.$$

Hence,

$$\beta^2 = 1 - \beta^2, \quad 2\beta^2 = 1,$$

so that

$$\beta = \frac{1}{\sqrt{2}}, \quad v = \frac{c}{\sqrt{2}}.$$

The positive root is taken because v denotes the speed.

Equivalently, the total relativistic energy is

$$E = \sqrt{p^2 c^2 + m^2 c^4} = \sqrt{m^2 c^4 + m^2 c^4} = \sqrt{2} mc^2,$$

and therefore the kinetic energy is

$$K = E - mc^2 = (\sqrt{2} - 1)mc^2.$$

The non-relativistic substitution $p = mv$ would incorrectly give $v = c$, where the non-relativistic momentum formula is invalid. The relativistic result $v = c/\sqrt{2} < c$ is physically admissible.

CONCLUSION:

The de Broglie wavelength of a massive particle equals its Compton wavelength when

$$p = mc.$$

In terms of the particle speed and energy, the equivalent conditions are

$$v = \frac{c}{\sqrt{2}}, \quad E = \sqrt{2} mc^2, \quad K = (\sqrt{2} - 1)mc^2.$$



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