



UPSC PHYSICS OPTIONAL for CSE & IFoS

ABHI PHYSICS

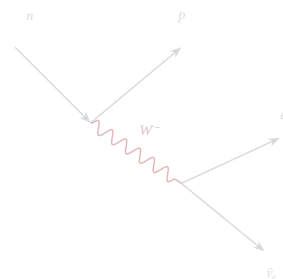
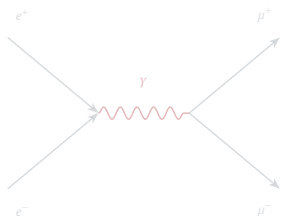
$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$E_n = -\frac{13.6}{n^2} \text{ eV}$$

PREVIOUS YEAR QUESTIONS



41 YEARS QUESTION BANK | 1986-2026

CSE + IFoS

SUBJECT

Nuclear and Particle Physics

10 SOLVED QUESTIONS • CSE + IFoS

$$n \rightarrow p + e^- + \bar{\nu}_e$$

$$E^2 = p^2 c^2 + m^2 c^4$$

$$\partial_\mu j^\mu = 0$$

SUBJECT-WISE • TOPIC-WISE • SOURCE-TAGGED

For Civil Services and Indian Forest Service Examinations

Sample Preview extbar 10 Solved Questions

Question Index

No.	Question	Marks	Year	Exam	Page
1	Are the nucleons inside the nucleus governed by the laws of quantum physics? Justify your answer.	10	2026	CSE	3
2	Binding energy and rest mass energy of a two-nucleon bound state are denoted by B and Mc^2 , respectively, where c is the speed of light. Calculate the minimum energy of a photon required to dissociate this bound state in terms of B and Mc^2 .	10	2026	CSE	4
3	A nucleus of rest mass M is initially in an excited state whose energy is ΔE above its ground state. The nucleus emits a γ -ray of energy $h\nu$ and makes a transition to its ground state. Calculate the fractional change in energy for the nucleus.	10	2026	CSE	5
4	By considering the three-dimensional harmonic oscillator potential, explain the energy levels of nucleons inside the nucleus. Also, derive the zero-point energy of nucleons. Why did this assumption fail to explain the existence of nuclei with higher magic number?	20	2026	CSE	6
5	What do you understand by the packing fraction ' f ' and the binding energy ' E_b ' of a nucleus? Draw the graphs for packing fraction f versus mass number (A) and binding energy fraction $f_b \left(= \frac{E_b}{A} \right)$ versus mass number (A). Further, explain how the graphs of the variation of f and f_b with A have complementary approaches.	10	2026	CSE	8
6	What is the criticality of a self-sustained nuclear reactor? Write the basic processes affecting the nuclear chain reactions of a finite-size nuclear reactor. Also, write the critical size of reactors of different shapes in terms of geometrical buckling constant.	20	2026	CSE	10
7	The total binding energies of $^{15}_8\text{O}$, $^{16}_8\text{O}$ and $^{17}_8\text{O}$ are $111 \cdot 96$ MeV, $127 \cdot 62$ MeV and $131 \cdot 76$ MeV respectively. Determine the energy gap between $1p_{1/2}$ and $1d_{5/2}$ neutron shells for the nuclide whose mass number is close to 16.	15	2025	CSE	12
8	What is the minimum energy required to break a ^4_2He nucleus into free protons and neutrons? [Given, $m_H = 1 \cdot 007825$ amu, $m_n = 1 \cdot 008665$ amu, $m_e = 0 \cdot 00055$ amu and $m_{\text{He}} = 4 \cdot 002603$ amu]	15	2025	CSE	13
9	Consider a uranium nucleus ($_{92}\text{U}^{236}$) breaking up spontaneously into two equal parts. Estimate the reduction of electrostatic energy of the nucleus considering uniform charge distribution. [Assume that nuclear radius is $1 \cdot 2 \times 10^{-13} A^{1/3}$ cm]	15	2025	CSE	14
10	State the basic assumption of single-particle shell model. How do the centrifugal and spin-orbit terms remove the degeneracy of three-dimensional spherical harmonic oscillator?	10+10=20	2025	CSE	15

CSE • 2026 • 10 MARKS

Q1. Are the nucleons inside the nucleus governed by the laws of quantum physics? Justify your answer.

INTRODUCTION:

Yes. Protons and neutrons inside a nucleus are microscopic particles confined to a region of linear dimension of order a few femtometres. Their motion must therefore be described by quantum mechanics rather than by classical trajectories. This can be justified quantitatively by comparing the nucleon's de Broglie wavelength with the nuclear size and is also confirmed by characteristic nuclear phenomena such as discrete energy levels, spin and parity, shell structure, tunnelling, and the Pauli exclusion principle.

Let a nucleon of mass m_N be localized within a nucleus of radius

$$R \simeq r_0 A^{1/3}, \quad r_0 \simeq 1.2 \text{ fm},$$

where A is the mass number and $1 \text{ fm} = 10^{-15} \text{ m}$.

SOLUTION:

Localization of a nucleon to a distance of order R implies, from the Heisenberg uncertainty principle,

$$\Delta r \Delta p \gtrsim \frac{\hbar}{2}.$$

Taking $\Delta r \sim R$, the characteristic momentum is at least of order

$$p \sim \frac{\hbar}{R}.$$

Consequently, the corresponding reduced de Broglie wavelength is

$$\frac{\lambda}{2\pi} = \frac{\hbar}{p} \sim R,$$

or equivalently,

$$\lambda \sim 2\pi R.$$

Thus the matter-wave length scale of a confined nucleon is comparable to, or larger than, the dimensions of the nucleus. The nucleon cannot be represented as a localized classical particle following a definite orbit; its wave character is essential throughout the nuclear volume.

The same conclusion follows from the characteristic confinement energy. In the non-relativistic approximation,

$$E_{\text{conf}} \sim \frac{p^2}{2m_N} \sim \frac{\hbar^2}{2m_N R^2}.$$

For a representative nuclear length $R \sim 2 \text{ fm}$, using

$$\hbar c \simeq 197 \text{ MeV fm}, \quad m_N c^2 \simeq 939 \text{ MeV},$$

one obtains

$$E_{\text{conf}} \sim \frac{(\hbar c)^2}{2(m_N c^2) R^2} = \frac{(197 \text{ MeV fm})^2}{2(939 \text{ MeV})(2 \text{ fm})^2} \simeq 5.2 \text{ MeV}.$$

This energy scale is comparable to typical nuclear excitation and binding-energy scales, so quantum confinement is not a small correction to classical motion. It determines the nuclear states themselves. Moreover,

$$\frac{pc}{m_N c^2} \sim \frac{\hbar c}{R m_N c^2} \simeq \frac{197}{2 \times 939} \simeq 0.10,$$

so this illustrative estimate is consistent with a predominantly non-relativistic quantum treatment, although relativistic and subnucleonic corrections may be required for greater precision.

Several observed properties of nuclei provide direct evidence:

1. Nuclei possess discrete energy levels. Their emission and absorption of γ -rays occur at definite energies

$$E_\gamma = E_i - E_f,$$

as expected for transitions between quantized stationary states.

2. Protons and neutrons are spin- $\frac{1}{2}$ fermions. Identical nucleons obey the Pauli exclusion principle, so they fill nuclear quantum states rather than all collapsing into the lowest state. This produces Fermi motion even in the nuclear ground state.
3. Nuclear states have definite angular momentum and parity. The observed shell structure and the enhanced stability of nuclei having magic numbers are consequences of quantized single-particle states in an effective nuclear potential.
4. Processes such as α -decay occur by quantum tunnelling through the Coulomb barrier. Classically, an α -particle whose energy is below the barrier could not escape, whereas quantum mechanics gives a finite penetration probability.
5. A nucleus is described by a many-nucleon wavefunction,

$$\Psi(\mathbf{r}_1, \dots, \mathbf{r}_A; \sigma_1, \dots, \sigma_A),$$

which evolves according to the quantum Hamiltonian,

$$\hat{H}\Psi = E\Psi$$

for a stationary state. The wavefunction must have the appropriate antisymmetry under interchange of identical protons or identical neutrons.

The uncertainty-principle estimate also gives the correct classical limit: if the confining length R were made very large compared with the reduced de Broglie wavelength, then $\hbar/(pR) \ll 1$, quantum interference would become less important, and a classical description could emerge. In an actual nucleus, however, $\hbar/(pR)$ is of order unity, so that limit is not applicable.

CONCLUSION:

The nucleons inside a nucleus are necessarily governed by quantum physics. Their de Broglie wavelength is comparable to the nuclear dimensions, and confinement produces momenta and energies of nuclear magnitude. Hence nucleons do not follow definite classical trajectories; they occupy quantized many-body states characterized by energy, angular momentum, spin and parity. Discrete spectra, nuclear shell structure, Fermi motion, the Pauli exclusion principle and quantum tunnelling provide experimental manifestations of this quantum behaviour.

CSE • 2026 • 10 MARKS

Q2. Binding energy and rest mass energy of a two-nucleon bound state are denoted by B and Mc^2 , respectively, where c is the speed of light. Calculate the minimum energy of a photon required to dissociate this bound state in terms of B and Mc^2 .

INTRODUCTION:

Consider the bound state initially at rest in the laboratory frame. Let m_1 and m_2 be the rest masses of the two free nucleons. By definition of the binding energy,

$$B = (m_1 + m_2 - M)c^2$$

so that

$$(m_1 + m_2)c^2 = Mc^2 + B.$$

The photon is taken to propagate along the positive z -direction. Four-momenta are written as

$$p^\mu = \left(\frac{E}{c}, \mathbf{p} \right)$$

with metric signature $(+, -, -, -)$, so that $p^\mu p_\mu = m^2 c^2$. Energy and momentum conservation are most conveniently imposed through the invariant mass of the complete initial system. At threshold, the two outgoing nucleons have zero relative momentum in their centre-of-momentum frame, which minimizes their total invariant energy.

SOLUTION:

If E_γ is the incident photon energy, the four-momenta of the stationary target and the photon are

$$P^\mu = (Mc, \mathbf{0}), \quad k^\mu = \left(\frac{E_\gamma}{c}, 0, 0, \frac{E_\gamma}{c} \right).$$

Since the photon is massless,

$$k^2 = 0,$$

and

$$P \cdot k = (Mc) \left(\frac{E_\gamma}{c} \right) = ME_\gamma.$$

Therefore, the squared total initial four-momentum is

$$(P + k)^2 = P^2 + 2P \cdot k + k^2 = M^2 c^2 + 2ME_\gamma.$$

In the centre-of-momentum frame of the final nucleons, their momenta are equal and opposite:

$$\mathbf{p}_1^* = \mathbf{q}, \quad \mathbf{p}_2^* = -\mathbf{q}.$$

Their total energy in that frame is

$$E_f^*(q) = \sqrt{m_1^2 c^4 + q^2 c^2} + \sqrt{m_2^2 c^4 + q^2 c^2}.$$

This is minimized at $q = 0$. Hence, at the dissociation threshold,

$$E_{f,\text{th}}^* = (m_1 + m_2)c^2,$$

and the squared final total four-momentum is

$$(P_f)^2 = (m_1 + m_2)^2 c^2.$$

Four-momentum conservation gives $P + k = P_f$, so at threshold

$$M^2 c^2 + 2ME_{\gamma,\text{min}} = (m_1 + m_2)^2 c^2.$$

Multiplying by c^2 and using

$$(m_1 + m_2)c^2 = Mc^2 + B,$$

we obtain

$$M^2 c^4 + 2Mc^2 E_{\gamma,\text{min}} = (Mc^2 + B)^2.$$

Thus,

$$\begin{aligned} 2Mc^2 E_{\gamma,\text{min}} &= (Mc^2 + B)^2 - M^2 c^4 \\ &= 2Mc^2 B + B^2, \end{aligned}$$

which yields

$$E_{\gamma,\text{min}} = B + \frac{B^2}{2Mc^2} = B \left(1 + \frac{B}{2Mc^2} \right).$$

The threshold exceeds B because a photon carries momentum E_γ/c . Since the target is initially at rest, the final two-nucleon system must carry this momentum in the positive z -direction even when its internal relative kinetic energy is zero. For weak binding, $B \ll Mc^2$,

$$E_{\gamma,\text{min}} = B \left(1 + \frac{B}{2Mc^2} \right) \simeq B,$$

with the small positive recoil correction $B^2/(2Mc^2)$.

CONCLUSION:

For an isolated two-nucleon bound state of rest energy Mc^2 , initially at rest, the minimum photon energy required for dissociation is

$$E_{\gamma, \min} = B \left(1 + \frac{B}{2Mc^2} \right).$$

Here $E_{\gamma, \min}$, B , and Mc^2 are all measured in energy units. At threshold the two nucleons have zero relative momentum in their centre-of-momentum frame, while their total laboratory momentum is directed along the incident photon.

CSE • 2026 • 10 MARKS

Q3. A nucleus of rest mass M is initially in an excited state whose energy is ΔE above its ground state. The nucleus emits a γ -ray of energy $h\nu$ and makes a transition to its ground state. Calculate the fractional change in energy for the nucleus.

INTRODUCTION:

The emitted photon carries momentum

$$p_{\gamma} = \frac{h\nu}{c}.$$

Since the initial nucleus is at rest, conservation of momentum requires the ground-state nucleus to recoil with equal and opposite momentum. Consequently, the photon energy is slightly smaller than the excitation energy ΔE .

Let M denote the rest mass of the ground-state nucleus. The initial energy is then

$$E_i = Mc^2 + \Delta E.$$

If $E_{\gamma} = h\nu$ is the photon energy, the recoiling nucleus has momentum magnitude $p = E_{\gamma}/c$ and relativistic energy

$$E_N = \sqrt{M^2c^4 + p^2c^2}.$$

SOLUTION:

Energy and momentum conservation give

$$Mc^2 + \Delta E = E_{\gamma} + \sqrt{M^2c^4 + E_{\gamma}^2}.$$

Therefore,

$$Mc^2 + \Delta E - E_{\gamma} = \sqrt{M^2c^4 + E_{\gamma}^2}.$$

Squaring both sides,

$$(Mc^2 + \Delta E)^2 - 2E_{\gamma}(Mc^2 + \Delta E) + E_{\gamma}^2 = M^2c^4 + E_{\gamma}^2.$$

After cancellation,

$$2Mc^2\Delta E + (\Delta E)^2 = 2E_{\gamma}(Mc^2 + \Delta E),$$

so that

$$E_{\gamma} = h\nu = \frac{\Delta E(2Mc^2 + \Delta E)}{2(Mc^2 + \Delta E)}.$$

Equivalently,

$$h\nu = \Delta E \left(1 - \frac{\Delta E}{2(Mc^2 + \Delta E)} \right).$$

The energy retained by the nucleus as recoil kinetic energy is

$$T_R = \Delta E - h\nu = \frac{(\Delta E)^2}{2(Mc^2 + \Delta E)}.$$

Hence the fractional change, i.e. the fraction of the transition energy going into nuclear recoil, is

$$\frac{\Delta E - h\nu}{\Delta E} = \frac{\Delta E}{2(Mc^2 + \Delta E)}.$$

For ordinary nuclear transitions,

$$\Delta E \ll Mc^2,$$

and therefore

$$\frac{\Delta E - h\nu}{\Delta E} \simeq \frac{\Delta E}{2Mc^2}.$$

In the same approximation,

$$T_R \simeq \frac{(\Delta E)^2}{2Mc^2}, \quad h\nu \simeq \Delta E \left(1 - \frac{\Delta E}{2Mc^2} \right).$$

This agrees with the non-relativistic recoil expression

$$T_R = \frac{p^2}{2M} = \frac{(h\nu)^2}{2Mc^2} \simeq \frac{(\Delta E)^2}{2Mc^2}.$$

CONCLUSION:

The photon energy is reduced below the excitation energy because the ground-state nucleus must recoil. The fractional energy change is

$$\frac{\Delta E - h\nu}{\Delta E} = \frac{\Delta E}{2(Mc^2 + \Delta E)} \simeq \frac{\Delta E}{2Mc^2} \quad (\Delta E \ll Mc^2).$$

Thus the nucleus receives the recoil energy

$$T_R \simeq \frac{(\Delta E)^2}{2Mc^2}.$$

in the direction opposite to the emitted γ -ray, while

$$h\nu \simeq \Delta E \left(1 - \frac{\Delta E}{2Mc^2} \right).$$

CSE • 2026 • 20 MARKS

Q4. By considering the three-dimensional harmonic oscillator potential, explain the energy levels of nucleons inside the nucleus. Also, derive the zero-point energy of nucleons. Why did this assumption fail to explain the existence of nuclei with higher magic number?

INTRODUCTION:

In the independent-particle shell model, each nucleon is assumed to move in an average, spherically symmetric potential produced by all the other nucleons. Near the centre of a smooth potential well, its leading radial dependence is approximately quadratic. We therefore take

$$V(r) = -V_0 + \frac{1}{2}m\omega^2 r^2,$$

where m is the nucleon mass, ω is the oscillator frequency, and $-V_0$ is the value of the mean potential at its minimum. The coordinate $\mathbf{r}$ is measured from the nuclear centre, and the single-particle energy is measured relative to the zero chosen outside the potential. Thus, the excitation energy above the bottom of the well is independent of V_0 .

The assumptions are that the nucleons move independently, the mean field is central and identical in form for all nucleons of a given species, and residual interactions, Coulomb corrections and spin-orbit coupling are initially neglected. Protons and neutrons fill their respective levels separately according to the Pauli exclusion principle. Each spatial state can contain two identical nucleons because the nucleon has spin $s = \frac{1}{2}$.

SOLUTION:

1. The three-dimensional oscillator spectrum

The single-particle Hamiltonian is

$$H = -V_0 + \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2 r^2.$$

Since $r^2 = x^2 + y^2 + z^2$ and $\mathbf{p}^2 = p_x^2 + p_y^2 + p_z^2$, it separates into three one-dimensional oscillators:

$$H = -V_0 + \sum_{i=x,y,z} \left(\frac{p_i^2}{2m} + \frac{1}{2}m\omega^2 x_i^2 \right).$$

The energy eigenvalues are consequently

$$\epsilon_{n_x n_y n_z} = -V_0 + \hbar\omega \left(n_x + n_y + n_z + \frac{3}{2} \right) \\ n_x, n_y, n_z = 0, 1, 2, \dots$$

Introducing the principal oscillator quantum number

$$N = n_x + n_y + n_z,$$

one obtains

$$\epsilon_N = -V_0 + \left(N + \frac{3}{2} \right) \hbar\omega.$$

Thus all states with the same value of N are degenerate in the pure oscillator model. In spherical coordinates,

$$N = 2n_r + l,$$

where $n_r = 0, 1, 2, \dots$ is the radial quantum number and l is the orbital angular-momentum quantum number. Hence

$$\epsilon_{n_r l} = -V_0 + \left(2n_r + l + \frac{3}{2} \right) \hbar\omega.$$

For a given N , the allowed values are

$$l = N, N-2, N-4, \dots,$$

ending at $l = 0$ or $l = 1$. The parity of every state in the N -th oscillator shell is

$$\pi = (-1)^l = (-1)^N.$$

The number of non-negative integer triples satisfying $n_x + n_y + n_z = N$ is

$$g_N^{(\text{spatial})} = \binom{N+2}{2} = \frac{(N+1)(N+2)}{2}.$$

Including the two possible spin projections, the capacity of the N -th shell for protons or neutrons separately is

$$g_N = 2g_N^{(\text{spatial})} = (N+1)(N+2).$$

Therefore the shell capacities and cumulative occupation numbers are

N	orbitals in the shell	g_N	$\sum_{n=0}^N g_n$
0	1s	2	2
1	1p	6	8
2	1d, 2s	12	20
3	1f, 2p	20	40
4	1g, 2d, 3s	30	70
5	1h, 2f, 3p	42	112

Indeed,

$$\sum_{n=0}^N (n+1)(n+2) = \frac{(N+1)(N+2)(N+3)}{3}.$$

The pure three-dimensional harmonic oscillator therefore predicts the shell-closure sequence

$$\boxed{2, 8, 20, 40, 70, 112, \dots}$$

for neutron number or proton number separately. It correctly accounts for the first three observed magic numbers 2, 8, and 20.

2. Derivation of the zero-point energy

For each Cartesian coordinate, define the annihilation and creation operators

$$a_i = \sqrt{\frac{m\omega}{2\hbar}} x_i + \frac{i}{\sqrt{2m\hbar\omega}} p_i$$

$$a_i^\dagger = \sqrt{\frac{m\omega}{2\hbar}} x_i - \frac{i}{\sqrt{2m\hbar\omega}} p_i.$$

Using $[x_i, p_j] = i\hbar\delta_{ij}$, one finds

$$[a_i, a_j^\dagger] = \delta_{ij}$$

and

$$\frac{p_i^2}{2m} + \frac{1}{2}m\omega^2 x_i^2 = \hbar\omega \left(a_i^\dagger a_i + \frac{1}{2} \right).$$

Consequently,

$$H + V_0 = \hbar\omega \sum_{i=x,y,z} \left(a_i^\dagger a_i + \frac{1}{2} \right).$$

Since the eigenvalues of $a_i^\dagger a_i$ are the non-negative integers n_i , the smallest possible oscillator energy occurs for

$$n_x = n_y = n_z = 0.$$

It is

$$E_{zp} = \hbar\omega \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) = \boxed{\frac{3}{2}\hbar\omega}.$$

Thus the lowest single-particle energy, including the depth of the nuclear mean field, is

$$\boxed{\epsilon_0 = -V_0 + \frac{3}{2}\hbar\omega}.$$

The quantity $\frac{3}{2}\hbar\omega$ is the zero-point energy above the bottom of the well. It cannot be removed while retaining the oscillator potential because simultaneous vanishing of all three coordinate and momentum uncertainties would violate the uncertainty principle.

For A independent occupied single-particle states, the oscillator contribution to the total energy is

$$E_{osc} = \sum_{i=1}^A \left(N_i + \frac{3}{2} \right) \hbar\omega = \hbar\omega \sum_{i=1}^A N_i + \frac{3A}{2} \hbar\omega.$$

The Pauli principle prevents all identical nucleons from occupying $N_i = 0$, so the total energy also contains the shell-filling contribution $\hbar\omega \sum_i N_i$. A commonly used phenomenological nuclear estimate is

$$\hbar\omega \simeq 41A^{-1/3} \text{ MeV},$$

which gives the oscillator level spacing and makes the single-nucleon zero-point energy approximately

$$E_{zp} \simeq 61.5A^{-1/3} \text{ MeV}.$$

3. Failure for the higher magic numbers

The experimentally established higher magic numbers are

$$28, \quad 50, \quad 82, \quad 126,$$

whereas the pure oscillator gives 40, 70, 112, ... The failure follows from the excessive degeneracy of the harmonic oscillator. Its energy depends only on

$$N = 2n_r + l$$

and is independent of both l and the total angular momentum

$$\mathbf{j} = \mathbf{l} + \mathbf{s}.$$

Hence the two states

$$j = l + \frac{1}{2}, \quad j = l - \frac{1}{2}$$

are predicted to have the same energy.

In an actual nucleus there is a strong spin-orbit interaction,

$$V_{ls}(r) \mathbf{l} \cdot \mathbf{s},$$

and

$$\mathbf{l} \cdot \mathbf{s} = \frac{1}{2} [j(j+1) - l(l+1) - s(s+1)] \hbar^2.$$

For $s = \frac{1}{2}$,

$$\mathbf{l} \cdot \mathbf{s} = \begin{cases} \frac{l}{2} \hbar^2, & j = l + \frac{1}{2}, \\ -\frac{l+1}{2} \hbar^2, & j = l - \frac{1}{2}. \end{cases}$$

The nuclear spin-orbit interaction has the observed ordering in which the aligned state $j =$

$l + \frac{1}{2}$ is lowered substantially relative to the $j = l - \frac{1}{2}$ state. The splitting grows with l , so high- l orbitals may be displaced across the gap between adjacent oscillator shells.

For example, immediately above the closure at 20, the $1f_{7/2}$ level is pulled down. Its degeneracy is

$$2j + 1 = 2\left(\frac{7}{2}\right) + 1 = 8.$$

Filling this level therefore gives

$$20 + 8 = 28,$$

producing the observed magic number 28 instead of the oscillator closure at 40. Analogous rearrangements involving the $1g_{9/2}$, $1h_{11/2}$, and $1i_{13/2}$ orbitals lead to the closures at 50, 82, and 126, respectively.

The finite nuclear surface also makes the realistic mean field closer to a Woods-Saxon potential than to an oscillator at large r , thereby removing part of the accidental N -shell degeneracy. Nevertheless, the strong spin-orbit interaction is the decisive ingredient required to reproduce the higher magic numbers.

CONCLUSION:

For a nucleon moving in the spherical harmonic-oscillator mean field,

$$V(r) = -V_0 + \frac{1}{2}m\omega^2 r^2,$$

the single-particle energies are

$$\varepsilon_N = -V_0 + \left(N + \frac{3}{2}\right)\hbar\omega$$

$$N = n_x + n_y + n_z = 2n_r + l.$$

The zero-point energy per nucleon, measured upward from the potential minimum, is

$$E_{zp} = \frac{3}{2}\hbar\omega,$$

or approximately $61.5A^{-1/3}$ MeV when the phenomenological estimate $\hbar\omega \simeq 41A^{-1/3}$ MeV is used.

Including spin, the N -th oscillator shell can contain

$$g_N = (N + 1)(N + 2)$$

protons or neutrons separately, giving the predicted closures

$$2, 8, 20, 40, 70, 112, \dots$$

Thus the oscillator approximation is useful near the centre of a spherical nuclear mean field and explains the first three magic numbers, but it fails for higher

shells because it neglects the finite-surface form of the potential and, most importantly, the strong spin-orbit splitting. The lowering of the aligned $j = l + \frac{1}{2}$ orbitals reorganizes the spectrum and produces the observed higher magic numbers

$$28, 50, 82, 126.$$

CSE · 2026 · 10 MARKS

Q5. What do you understand by the packing fraction ' f ' and the binding energy ' E_b ' of a nucleus? Draw the graphs for packing fraction f versus mass number (A) and binding energy fraction $f_b \left(= \frac{E_b}{A} \right)$ versus mass number (A). Further, explain how the graphs of the variation of f and f_b with A have complementary approaches.

INTRODUCTION:

The question requires the definitions of packing fraction and nuclear binding energy, their qualitative variations with mass number A , and the physical reason for the complementary nature of these variations.

Consider a nuclide containing Z protons and $N = A - Z$ neutrons. Let $M(A, Z)$ be its ground-state atomic mass, u the unified atomic mass unit, m_H the mass of a hydrogen atom, and m_n the neutron mass. The binding-energy sign convention adopted here is

$$E_b > 0$$

for a bound nucleus; thus E_b is the energy required to separate the nucleus completely into free nucleons at rest and at infinite separation. Atomic masses will be used, with electronic binding-energy corrections neglected because they are very small on the nuclear-energy scale.

Since different isobars with the same A have different masses, the smooth curves represent nuclei on or near the valley of stability. Shell, pairing, and odd-even effects produce local deviations from these average trends.

SOLUTION:

1. The packing fraction is the fractional excess or deficiency of the actual isotopic mass over the integral mass number, conventionally multiplied by 10^4 :

$$f = 10^4 \frac{M/u - A}{A} = 10^4 \frac{M - Au}{Au}.$$

It is dimensionless. Some authors omit the factor 10^4 ; that convention changes only the numerical scale of the vertical axis.

The sign convention is therefore

$$\begin{aligned} f &> 0 \\ \iff \\ M &> Au \\ f &< 0 \\ \iff \\ M &< Au. \end{aligned}$$

Thus a negative packing fraction means that the average atomic mass per nucleon is less than u .

2. The nuclear binding energy is the work required to disintegrate the nucleus into its Z free protons and N free neutrons. It is also the energy released when these free nucleons combine to form the nucleus. In terms of nuclear masses,

$$E_b = [Zm_p + Nm_n - m_{\text{nuc}}(A, Z)] c^2.$$

Using atomic masses and neglecting the small electronic binding-energy correction,

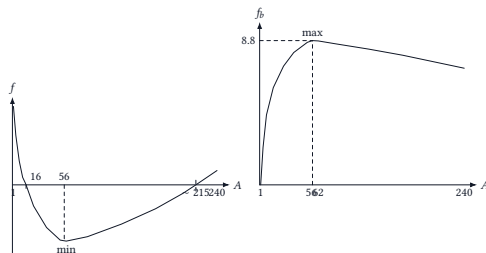
$$E_b \simeq [Zm_H + (A - Z)m_n - M(A, Z)] c^2.$$

The binding energy per nucleon, called the binding-energy fraction in the question, is

$$f_b = \frac{E_b}{A},$$

usually expressed in MeV nucleon^{-1} . It measures the average energy needed to remove a nucleon, although individual separation energies need not equal E_b/A .

3. The required qualitative graphs are shown below. The first zero crossing of the smooth packing-fraction curve occurs near $A \simeq 15$ –16; the light nuclei with $A \simeq 12$ –15 are therefore not to be represented on the negative branch of the smooth curve.



For the lightest nuclei, f is positive and falls rapidly with increasing A . Its smooth trend crosses the zero line near $A \simeq 15$ –16, becomes negative, and reaches a broad minimum

in the iron–nickel region. It then rises gradually and becomes positive again for sufficiently heavy nuclei, with the second crossing occurring roughly in the $A \simeq 210$ –220 region.

For a single free nucleon, $E_b = 0$, so $f_b = 0$. As nucleons form light nuclei, f_b rises rapidly. It attains a broad maximum of approximately

$$f_b \simeq 8.7\text{--}8.8 \text{ MeV nucleon}^{-1}$$

in the iron–nickel region, approximately $A \simeq 56$ –62, and then decreases slowly for heavier nuclei.

4. The complementary behavior follows directly from mass–energy conservation. The atomic-mass expression for binding energy gives

$$M(A, Z) \simeq Zm_H + Nm_n - \frac{E_b}{c^2}.$$

Subtracting Au , where $A = Z + N$, yields

$$M - Au \simeq Z(m_H - u) + N(m_n - u) - \frac{E_b}{c^2}.$$

Dividing by Au and using $f_b = E_b/A$,

$$\frac{M - Au}{Au} \simeq \frac{Z}{A} \left(\frac{m_H}{u} - 1 \right) + \frac{N}{A} \left(\frac{m_n}{u} - 1 \right) - \frac{f_b}{uc^2}.$$

Hence

$$\frac{f}{10^4} \simeq \frac{Z}{A} \left(\frac{m_H}{u} - 1 \right) + \frac{N}{A} \left(\frac{m_n}{u} - 1 \right) - \frac{f_b}{uc^2}.$$

The first two terms specify the mass per nucleon of the corresponding unbound proton–neutron mixture. Along the valley of stability they vary comparatively slowly with A , whereas the binding contribution enters with a negative sign. Greater binding energy per nucleon removes more rest mass per nucleon; consequently,

$$f_b \text{ increasing} \implies f \text{ decreasing},$$

and

$$f_b \text{ decreasing} \implies f \text{ increasing}.$$

Thus the broad maximum of f_b lies approximately in the same iron–nickel region in which f has its broad minimum.

The two graphs are complementary but are not exact mirror images. The reason is that the neutron-to-proton ratio, and therefore Z/A and N/A , changes systematically along the valley of stability. Shell structure, pairing, and odd–even effects also introduce local irregularities.

The maximum of f_b has an immediate energetic implication. Fusion of light nuclei can release energy when the product lies nearer the iron–nickel maximum and therefore has a larger E_b/A . Fission of a very heavy nucleus can likewise release energy when its medium-mass fragments have a larger E_b/A than the parent.

CONCLUSION:

For a nuclide of atomic mass $M(A, Z)$, the conventional packing fraction is

$$f = 10^4 \frac{M - Au}{Au},$$

and the positive binding energy and binding energy per nucleon are

$$E_b \approx [Zm_H + (A - Z)m_n - M(A, Z)]c^2$$

$$f_b = \frac{E_b}{A}.$$

Here f is dimensionless, E_b is an energy, and f_b is normally quoted in MeV nucleon⁻¹.

For nuclei near the valley of stability, the smooth $f(A)$ curve falls through zero near $A \approx 15$ – 16 , reaches a broad minimum near the iron–nickel region, and then rises, becoming positive again among very heavy nuclei. In the complementary variation, f_b rises from zero at $A = 1$ to approximately 8.7–8.8 MeV nucleon⁻¹ near $A \approx 56$ – 62 , and thereafter decreases slowly.

This complementarity results from the negative binding term in

$$\frac{f}{10^4} \approx \frac{Z}{A} \left(\frac{m_H}{u} - 1 \right) + \frac{N}{A} \left(\frac{m_n}{u} - 1 \right) - \frac{f_b}{uc^2}.$$

Subject to the gradual change of nuclear composition and local structure effects, a maximum of binding energy per nucleon therefore corresponds to a minimum of packing fraction. The iron–nickel region contains the most strongly bound nuclei on average, and motion toward this region permits energy release by fusion of light nuclei or fission of very heavy nuclei.

CSE • 2026 • 20 MARKS

Q6. What is the criticality of a self-sustained nuclear reactor? Write the basic processes affecting the nuclear chain reactions of a finite-size nuclear reactor. Also, write the critical size of reactors of different

shapes in terms of geometrical buckling constant.

INTRODUCTION:

A reactor is self-sustained when the neutron population reproduces itself from one generation to the next. Its criticality is therefore specified by the effective neutron multiplication factor

$$k_{\text{eff}} = \frac{\text{number of neutrons in one generation}}{\text{number of neutrons in the preceding generation}}.$$

Thus,

$$k_{\text{eff}} < 1$$

subcritical

$$k_{\text{eff}} = 1$$

critical

$$k_{\text{eff}} > 1$$

supercritical.

Equivalently, the reactivity is

$$\rho = \frac{k_{\text{eff}} - 1}{k_{\text{eff}}},$$

so that $\rho = 0$ at criticality.

For the size calculation, consider a bare, homogeneous reactor in the one-group diffusion approximation. Let $\phi(\mathbf{r})$ be the neutron flux, D the diffusion coefficient, Σ_a the macroscopic absorption cross-section, Σ_f the macroscopic fission cross-section, and ν the mean number of neutrons emitted per fission. The geometrical dimensions appearing in the diffusion boundary condition are extrapolated dimensions: the flux is taken to vanish a small extrapolation distance δ outside the physical surface.

SOLUTION:

The neutron life cycle in a finite reactor contains the following basic processes.

1. A neutron is absorbed in a fissile nucleus and produces fission. If ν neutrons are emitted per fission, this is the principal neutron-production process.
2. Some fast neutrons produce additional fissions before slowing down. This increases the neutron population by the fast-fission factor ϵ .
3. Elastic and inelastic scattering reduce neutron energies. In a thermal reactor, moderation carries the neutrons from fission energies to thermal energies, where the fission probability of fuels such as ^{235}U is large.
4. During slowing down, some neutrons undergo non-fission capture, particularly in resonance-energy regions. The resonance escape probability p is the fraction that reaches thermal energy without such absorption.

5. Thermal neutrons may be absorbed in the fuel, moderator, coolant, structural material, control material, or fission products. The thermal utilization factor f is the fraction of all thermal-neutron absorptions that occur in the fuel.
6. Absorption in the fuel may cause either fission or radiative capture. The reproduction factor η is the number of fission neutrons produced per thermal neutron absorbed in the fuel.
7. Since the reactor has finite dimensions, fast and thermal neutrons may escape through its boundary. If P_f and P_t are the fast and thermal non-leakage probabilities, respectively, leakage reduces the multiplication factor.

For a thermal reactor, these processes are summarized by the six-factor formula

$$k_{\text{eff}} = \eta \epsilon p f P_f P_t.$$

The first four factors give the infinite-medium multiplication factor,

$$k_{\infty} = \eta \epsilon p f,$$

whereas $P_f P_t < 1$ accounts for leakage from a finite reactor. A self-sustained steady chain reaction requires

$$\eta \epsilon p f P_f P_t = 1.$$

Delayed neutrons do not appreciably alter the static critical-size condition, but they are essential for the controllable time-dependent response of a reactor near criticality.

In one-group diffusion theory, the neutron balance eigenvalue equation is

$$-D\nabla^2\phi + \Sigma_a\phi = \frac{\nu\Sigma_f}{k_{\text{eff}}}\phi.$$

At exact criticality, $k_{\text{eff}} = 1$, and hence

$$-D\nabla^2\phi + \Sigma_a\phi = \nu\Sigma_f\phi.$$

Therefore,

$$\nabla^2\phi + B_m^2\phi = 0, \quad B_m^2 = \frac{\nu\Sigma_f - \Sigma_a}{D},$$

where B_m^2 is the material buckling. It measures the excess neutron production available to compensate leakage.

For a reactor of a specified shape, the fundamental spatial mode satisfies

$$\nabla^2\phi + B_g^2\phi = 0,$$

with zero flux at the extrapolated boundary. Here B_g^2 is the geometrical buckling, which measures leakage

imposed by the size and shape. The criticality condition is consequently

$$B_g^2 = B_m^2.$$

For a fixed material, a smaller reactor has larger B_g^2 and hence excessive leakage, whereas a larger reactor has smaller B_g^2 .

For the common reactor geometries, the critical dimensions follow as below.

1. For a slab of physical thickness T , infinite in the other two directions, let

$$T_e = T + 2\delta$$

be its extrapolated thickness. The fundamental flux is proportional to

$$\sin\left(\frac{\pi x}{T_e}\right),$$

and therefore

$$B_g^2 = \left(\frac{\pi}{T_e}\right)^2.$$

At criticality,

$$T_e = \frac{\pi}{B_m}, \quad T_c = \frac{\pi}{B_m} - 2\delta.$$

2. For a rectangular parallelepiped having physical side lengths a , b , and c , define

$$a_e = a + 2\delta, \quad b_e = b + 2\delta, \quad c_e = c + 2\delta.$$

Separation of variables gives the fundamental mode

$$\phi \propto \sin\left(\frac{\pi x}{a_e}\right) \sin\left(\frac{\pi y}{b_e}\right) \sin\left(\frac{\pi z}{c_e}\right),$$

so that

$$B_g^2 = \pi^2 \left(\frac{1}{a_e^2} + \frac{1}{b_e^2} + \frac{1}{c_e^2} \right).$$

The critical dimensions satisfy

$$B_m^2 = \pi^2 \left[\frac{1}{(a + 2\delta)^2} + \frac{1}{(b + 2\delta)^2} + \frac{1}{(c + 2\delta)^2} \right].$$

For a cube of physical side L ,

$$B_g^2 = \frac{3\pi^2}{(L + 2\delta)^2}, \quad L_c = \frac{\sqrt{3}\pi}{B_m} - 2\delta.$$

3. For a finite right circular cylinder of physical radius R and height H , take

$$R_e = R + \delta, \quad H_e = H + 2\delta.$$

The fundamental radial solution is $J_0(2.405 r/R_e)$, since 2.405 is the first positive zero of J_0 . Including the axial mode gives

$$B_g^2 = \left(\frac{2.405}{R_e}\right)^2 + \left(\frac{\pi}{H_e}\right)^2.$$

Thus its critical dimensions obey

$$B_m^2 = \left(\frac{2.405}{R + \delta}\right)^2 + \left(\frac{\pi}{H + 2\delta}\right)^2.$$

A unique pair (R_c, H_c) is obtained only after specifying the cylinder's aspect ratio or one of its dimensions.

4. For a sphere of physical radius R , let

$$R_e = R + \delta.$$

The finite spherically symmetric solution is

$$\phi(r) = C \frac{\sin(B_g r)}{r}.$$

Regularity at $r = 0$ selects this solution, and the extrapolated boundary condition $\phi(R_e) = 0$ gives

$$B_g R_e = \pi.$$

Hence

$$B_g^2 = \left(\frac{\pi}{R_e}\right)^2, \quad R_e = \frac{\pi}{B_m},$$

and the physical critical radius is

$$R_c = \frac{\pi}{B_m} - \delta.$$

Since B_m and B_g both have the dimension of inverse length, their equality consistently determines a critical length. The relations also show the expected physical limit: increasing a reactor dimension decreases B_g^2 , reduces neutron leakage, and moves the reactor toward supercriticality; decreasing the dimension increases leakage and moves it toward subcriticality.

CONCLUSION:

A self-sustained nuclear reactor is critical when

$$k_{\text{eff}} = 1, \quad \rho = 0,$$

so that neutron production exactly balances absorption and outward leakage. In life-cycle notation this is

$$\eta \epsilon p f P_f P_t = 1.$$

In the bare, homogeneous, one-group diffusion model, the equivalent critical-size condition is

$$B_g^2 = B_m^2 = \frac{\nu \Sigma_f - \Sigma_a}{D},$$

where B_g and B_m have units of inverse length.

Using extrapolated dimensions, the geometrical bucklings are

$$B_g^2 = \begin{cases} \left(\frac{\pi}{T_e}\right)^2, & \text{slab,} \\ \pi^2 \left(\frac{1}{a_e^2} + \frac{1}{b_e^2} + \frac{1}{c_e^2}\right), & \text{rectangular parallelepiped,} \\ \left(\frac{2.405}{R_e}\right)^2 + \left(\frac{\pi}{H_e}\right)^2, & \text{finite cylinder,} \\ \left(\frac{\pi}{R_e}\right)^2, & \text{sphere.} \end{cases}$$

Accordingly,

$$T_c = \frac{\pi}{B_m} - 2\delta$$

$$L_c = \frac{\sqrt{3}\pi}{B_m} - 2\delta$$

$$R_{c,\text{sphere}} = \frac{\pi}{B_m} - \delta$$

while the critical radius and height of a finite cylinder satisfy

$$\left(\frac{2.405}{R_c + \delta}\right)^2 + \left(\frac{\pi}{H_c + 2\delta}\right)^2 = B_m^2.$$

These results apply within diffusion theory to a bare homogeneous reactor; reflectors, heterogeneous cores, multigroup effects, and anisotropic leakage require the corresponding modified boundary conditions and neutron-balance equations.

CSE • 2025 • 15 MARKS

Q7. The total binding energies of ${}^{15}_8\text{O}$, ${}^{16}_8\text{O}$ and ${}^{17}_8\text{O}$ are 111.96 MeV, 127.62 MeV and 131.76 MeV respectively. Determine the energy gap between $1p_{1/2}$ and $1d_{5/2}$ neutron shells for the nuclide whose mass number is close to 16.

INTRODUCTION:

The proton number is fixed at $Z = 8$, while the neutron numbers of the three isotopes are

$$N({}^{15}\text{O}) = 7, \quad N({}^{16}\text{O}) = 8, \quad N({}^{17}\text{O}) = 9.$$

In the independent-particle shell model, ${}^{16}\text{O}$ is a doubly magic closed-shell nucleus. Its highest occupied

neutron orbit is $1p_{1/2}$, whereas the next neutron added to form ^{17}O occupies the $1d_{5/2}$ orbit.

Let $B(A)$ denote the positive total binding energy of ^AO . The one-neutron separation energy is

$$S_n(A) = B(A) - B(A - 1).$$

Neglecting changes in the mean nuclear field and residual-particle correlations among the three neighbouring isotopes, $S_n(^{16}\text{O})$ gives the binding of a neutron in the last occupied $1p_{1/2}$ orbit, while $S_n(^{17}\text{O})$ gives the binding of the added neutron in the $1d_{5/2}$ orbit.

Single-particle energies are measured relative to the neutron-emission threshold, so bound-state energies are negative:

$$\epsilon_{n\ell j} \approx -S_n.$$

The required shell gap is therefore

$$\Delta E = \epsilon(1d_{5/2}) - \epsilon(1p_{1/2}).$$

SOLUTION:

The energy required to remove the eighth neutron from ^{16}O , leaving ^{15}O , is

$$\begin{aligned} S_n(^{16}\text{O}) &= B(^{16}\text{O}) - B(^{15}\text{O}) \\ &= 127.62 \text{ MeV} - 111.96 \text{ MeV} \\ &= 15.66 \text{ MeV}. \end{aligned}$$

Thus, in the shell-model approximation,

$$\epsilon(1p_{1/2}) \approx -15.66 \text{ MeV}.$$

Similarly, the energy required to remove the ninth neutron from ^{17}O , leaving the closed-shell ^{16}O core, is

$$\begin{aligned} S_n(^{17}\text{O}) &= B(^{17}\text{O}) - B(^{16}\text{O}) \\ &= 131.76 \text{ MeV} - 127.62 \text{ MeV} \\ &= 4.14 \text{ MeV}. \end{aligned}$$

Hence,

$$\epsilon(1d_{5/2}) \approx -4.14 \text{ MeV}.$$

The separation between the two single-neutron levels is consequently

$$\begin{aligned} \Delta E &= \epsilon(1d_{5/2}) - \epsilon(1p_{1/2}) \\ &= (-4.14) - (-15.66) \text{ MeV} \\ &= 11.52 \text{ MeV}. \end{aligned}$$

Equivalently,

$$\Delta E = S_n(^{16}\text{O}) - S_n(^{17}\text{O}) = 2B(^{16}\text{O}) - B(^{15}\text{O}) - B(^{17}\text{O}).$$

The positive value is physically consistent: the occupied $1p_{1/2}$ neutron is more deeply bound than the $1d_{5/2}$ neutron lying above the $N = 8$ shell closure.

CONCLUSION:

The neutron-shell energy gap near mass number $A = 16$ is

$$\Delta E = \epsilon(1d_{5/2}) - \epsilon(1p_{1/2}) = 11.52 \text{ MeV}.$$

Thus the $1d_{5/2}$ neutron level lies approximately 11.52 MeV above the $1p_{1/2}$ level. This inference assumes that the neighbouring oxygen isotopes can be described as one-neutron particle or hole states about the closed-shell ^{16}O core, with changes of the mean field and residual correlation energies neglected.

CSE · 2025 · 15 MARKS

Q8. What is the minimum energy required to break a ^4_2He nucleus into free protons and neutrons? [Given, $m_{\text{H}} = 1.007825 \text{ amu}$, $m_n = 1.008665 \text{ amu}$, $m_e = 0.00055 \text{ amu}$ and $m_{\text{He}} = 4.002603 \text{ amu}$]

INTRODUCTION:

The ^4_2He nucleus contains two protons and two neutrons. The minimum energy required to separate it into free nucleons is its nuclear binding energy. It is obtained from the mass defect,

$$\Delta m = (2m_p + 2m_n) - m_{^4\text{He nucleus}},$$

using Einstein's mass-energy relation,

$$E_{\text{min}} = \Delta m c^2.$$

The tabulated masses m_{H} and m_{He} are atomic masses. Hence

$$m_p = m_{\text{H}} - m_e, \quad m_{^4\text{He nucleus}} = m_{\text{He}} - 2m_e,$$

where the negligible atomic electron binding energies are ignored.

SOLUTION:

Substituting the nuclear masses into the mass-defect expression gives

$$\begin{aligned} \Delta m &= 2(m_{\text{H}} - m_e) + 2m_n - (m_{\text{He}} - 2m_e) \\ &= 2m_{\text{H}} + 2m_n - m_{\text{He}}. \end{aligned}$$

Thus the electron masses cancel, as expected because both sides contain two electrons when atomic masses are used consistently.

Numerically,

$$\begin{aligned} \Delta m &= [2(1.007825) + 2(1.008665) - 4.002603] \text{ amu} \\ &= [2.015650 + 2.017330 - 4.002603] \text{ amu} \\ &= 0.030377 \text{ amu}. \end{aligned}$$

Using

$$1 \text{ amu } c^2 \approx 931.5 \text{ MeV},$$

the required energy is

$$E_{\min} = (0.030377 \text{ amu})(931.5 \text{ MeV/amu}) \\ \approx 28.30 \text{ MeV}.$$

Equivalently, since $1 \text{ MeV} = 1.602 \times 10^{-13} \text{ J}$,

$$E_{\min} = (28.30)(1.602 \times 10^{-13}) \text{ J} \\ \approx 4.53 \times 10^{-12} \text{ J}.$$

The positive mass defect confirms that energy must be supplied to overcome the binding of the nucleons. This value also corresponds to an average binding energy of

$$\frac{28.30}{4} \approx 7.07 \text{ MeV}$$

per nucleon, consistent with the strong binding of the ${}^4\text{He}$ nucleus.

CONCLUSION:

The minimum energy required to break one ${}^4_2\text{He}$ nucleus completely into two free protons and two free neutrons is

$$E_{\min} \approx 28.3 \text{ MeV} \approx 4.53 \times 10^{-12} \text{ J}.$$

This is the total binding energy of the ${}^4_2\text{He}$ nucleus and is the threshold energy, neglecting recoil and atomic electron-binding corrections.

CSE • 2025 • 15 MARKS

Q9. Consider a uranium nucleus (${}_{92}\text{U}^{236}$) breaking up spontaneously into two equal parts. Estimate the reduction of electrostatic energy of the nucleus considering uniform charge distribution. [Assume that nuclear radius is $1.2 \times 10^{-13} A^{1/3} \text{ cm}$]

INTRODUCTION:

The electrostatic self-energy of a uniformly charged spherical nucleus of charge $Q = Ze$ and radius R is

$$E_C = \frac{3}{5} \frac{1}{4\pi\epsilon_0} \frac{Q^2}{R}.$$

We compare the self-energy of the original spherical nucleus with the sum of the self-energies of two equal spherical fragments after they are sufficiently separated that their mutual electrostatic interaction is negligible.

For the parent nucleus,

$$A = 236, \quad Z = 92, \quad Q = 92e.$$

Equal division, with the proton-to-nucleon ratio unchanged, gives

$$A_f = \frac{236}{2} = 118, \quad Z_f = \frac{92}{2} = 46,$$

so charge and nucleon number are conserved. Writing the parent and fragment radii as R and R_f , respectively, the given radius law implies

$$R = r_0 A^{1/3}, \quad R_f = r_0 \left(\frac{A}{2}\right)^{1/3} = \frac{R}{2^{1/3}},$$

where

$$r_0 = 1.2 \times 10^{-13} \text{ cm} = 1.2 \text{ fm}.$$

SOLUTION:

The radius of the original uranium nucleus is

$$R = (1.2 \text{ fm})(236)^{1/3}.$$

Since

$$236^{1/3} \approx 6.18,$$

we obtain

$$R \approx (1.2)(6.18) \text{ fm} \approx 7.42 \text{ fm}.$$

Using

$$\frac{e^2}{4\pi\epsilon_0} = 1.44 \text{ MeV fm},$$

the electrostatic self-energy of the parent nucleus is

$$E_C^{(i)} = \frac{3}{5} \frac{1}{4\pi\epsilon_0} \frac{Z^2 e^2}{R}.$$

Therefore,

$$E_C^{(i)} = \frac{3}{5} \frac{(92)^2 (1.44 \text{ MeV fm})}{7.42 \text{ fm}}.$$

Numerically,

$$E_C^{(i)} = \frac{0.6 \times 8464 \times 1.44}{7.42} \text{ MeV} \approx 9.86 \times 10^2 \text{ MeV}.$$

Each fragment has charge $Ze/2$ and radius $R/2^{1/3}$. Its self-energy is

$$E_{C,f} = \frac{3}{5} \frac{1}{4\pi\epsilon_0} \frac{(Ze/2)^2}{R/2^{1/3}}.$$

The total self-energy of both fragments is consequently

$$E_C^{(f)} = 2E_{C,f} \\ = 2 \left[\frac{3}{5} \frac{1}{4\pi\epsilon_0} \frac{Z^2 e^2 / 4}{R/2^{1/3}} \right] \\ = \frac{3}{5} \frac{1}{4\pi\epsilon_0} \frac{Z^2 e^2}{R} 2^{-2/3} \\ = E_C^{(i)} 2^{-2/3}.$$

Since

$$2^{-2/3} \approx 0.630,$$

the final electrostatic self-energy is

$$E_C^{(f)} \approx (9.86 \times 10^2)(0.630) \text{ MeV} \approx 6.21 \times 10^2 \text{ MeV}.$$

Hence the reduction in electrostatic energy is

$$\begin{aligned} \Delta E_C &= E_C^{(i)} - E_C^{(f)} \\ &= E_C^{(i)} \left(1 - 2^{-2/3}\right) \\ &\approx (9.86 \times 10^2)(1 - 0.630) \text{ MeV} \\ &\approx 3.65 \times 10^2 \text{ MeV}. \end{aligned}$$

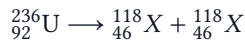
In SI units,

$$\Delta E_C \approx (3.65 \times 10^2)(1.602 \times 10^{-13}) \text{ J} \approx 5.85 \times 10^{-11} \text{ J}.$$

The positive value is physically consistent: division spreads the same total charge over two separated nuclei, thereby lowering the total electrostatic self-energy.

CONCLUSION:

For uniform spherical charge distributions, the spontaneous equal division



reduces the electrostatic self-energy by

$$\Delta E_C \approx 3.65 \times 10^2 \text{ MeV}$$

or equivalently

$$\Delta E_C \approx 5.85 \times 10^{-11} \text{ J}.$$

Thus approximately 365 MeV of electrostatic energy is released in the direction of fragment separation, under the assumptions of equal charge division, spherical uniform-charge fragments, unchanged nuclear density, and negligible mutual Coulomb interaction in the final sufficiently separated state.

CSE • 2025 • 10+10=20 MARKS

Q10. State the basic assumption of single-particle shell model. How do the centrifugal and spin-orbit terms remove the degeneracy of three-dimensional spherical harmonic oscillator?

INTRODUCTION:

The single-particle shell model assumes that, to a good approximation, each nucleon moves independently in a time-independent average potential produced by all the other nucleons. The complicated

residual two-nucleon interactions are replaced at leading order by a common, spherically symmetric mean field. The Pauli exclusion principle is imposed separately on protons and neutrons, and the residual interaction is treated as a comparatively small correction.

For a nucleon of mass m , the single-particle Hamiltonian may be written as

$$H = \frac{\mathbf{p}^2}{2m} + V(r) + V_{ls}(r) \mathbf{l} \cdot \mathbf{s}.$$

For the ideal three-dimensional isotropic harmonic oscillator,

$$\begin{aligned} V(r) &= \frac{1}{2} m \omega^2 r^2 \\ E_N^{(0)} &= \left(N + \frac{3}{2}\right) \hbar \omega \\ N &= 2n_r + l \end{aligned}$$

where $n_r = 0, 1, 2, \dots$ is the radial quantum number and l is the orbital angular-momentum quantum number. Thus all states having the same oscillator quantum number N , but different allowed values

$$l = N, N-2, \dots,$$

have the same energy. Including spin, a major oscillator shell has degeneracy

$$g_N = 2 \sum_l (2l+1) = (N+1)(N+2).$$

The centrifugal contribution is examined in the radial equation, while the spin-orbit interaction is evaluated in the coupled basis

$$\mathbf{j} = \mathbf{l} + \mathbf{s}, \quad s = \frac{1}{2}.$$

SOLUTION:

For a central potential, write the wavefunction as

$$\psi_{nljm}(\mathbf{r}, \sigma) = \frac{u_{nl}(r)}{r} \mathcal{Y}_{ljm}(\hat{\mathbf{r}}, \sigma).$$

The radial Schrödinger equation is

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + V(r) + \frac{\hbar^2 l(l+1)}{2mr^2} \right] u_{nl}(r) = E_{nl} u_{nl}(r).$$

Hence the effective radial potential is

$$V_{\text{eff}}(r) = V(r) + V_{\text{cf}}(r), \quad V_{\text{cf}}(r) = \frac{\hbar^2 l(l+1)}{2mr^2}.$$

The centrifugal barrier vanishes for $l = 0$ and increases with $l(l+1)$. Consequently, states with different l experience different effective radial potentials and have different radial distributions. In a general

nuclear mean field this makes the energy depend separately on n_r and l :

$$E_{n_r l} \neq E_{n_r l'} \quad \text{for } l \neq l',$$

even when $2n_r + l = 2n_r' + l'$. The accidental degeneracy between different l -values within one oscillator shell is therefore removed.

There is an important qualification. The centrifugal term is already present in the exact radial solution of the isotropic harmonic oscillator. For the special potential $V(r) = m\omega^2 r^2/2$, its l -dependence is exactly compensated by the radial excitation energy, giving

$$E_{n_r l}^{(0)} = \left(2n_r + l + \frac{3}{2}\right)\hbar\omega = \left(N + \frac{3}{2}\right)\hbar\omega.$$

Thus the centrifugal barrier by itself does not split an *exact* isotropic oscillator shell. The l -splitting arises when the oscillator is replaced by, or corrected toward, a realistic finite-depth nuclear mean field. It is often represented phenomenologically within a major shell by an l -dependent term such as

$$\Delta H_l = D l^2,$$

where the coefficient D , including its sign, depends on the chosen mean-field parametrization. Its eigenvalue is

$$\Delta E_l = D \hbar^2 l(l+1),$$

so states with different l are no longer degenerate. Equivalently, the centrifugal barrier in the radial equation supplies the physical origin of the sensitivity to l , while the non-oscillator form of the central potential prevents the exact oscillator cancellation.

The remaining degeneracy between the two possible orientations of spin relative to the orbital angular momentum is removed by the spin-orbit interaction. Let

$$H_{ls} = V_{ls}(r) \mathbf{l} \cdot \mathbf{s}.$$

Using

$$\mathbf{j}^2 = (\mathbf{l} + \mathbf{s})^2 = \mathbf{l}^2 + \mathbf{s}^2 + 2\mathbf{l} \cdot \mathbf{s},$$

one obtains

$$\mathbf{l} \cdot \mathbf{s} = \frac{1}{2} (\mathbf{j}^2 - \mathbf{l}^2 - \mathbf{s}^2).$$

Therefore, in a state labelled by l, s, j ,

$$\langle \mathbf{l} \cdot \mathbf{s} \rangle = \frac{\hbar^2}{2} [j(j+1) - l(l+1) - s(s+1)].$$

For a nucleon, $s = \frac{1}{2}$, so $s(s+1) = 3/4$, and the allowed total angular momenta are

$$j = l + \frac{1}{2} \quad \text{and} \quad j = l - \frac{1}{2} \quad (l \neq 0).$$

For $j = l + \frac{1}{2}$,

$$j(j+1) = \left(l + \frac{1}{2}\right)\left(l + \frac{3}{2}\right) = l(l+1) + l + \frac{3}{4},$$

and hence

$$\langle \mathbf{l} \cdot \mathbf{s} \rangle_{j=l+1/2} = \frac{\hbar^2 l}{2}.$$

For $j = l - \frac{1}{2}$,

$$j(j+1) = \left(l - \frac{1}{2}\right)\left(l + \frac{1}{2}\right) = l^2 - \frac{1}{4},$$

which gives

$$\langle \mathbf{l} \cdot \mathbf{s} \rangle_{j=l-1/2} = -\frac{\hbar^2(l+1)}{2}.$$

If the radial dependence is replaced by its expectation value in the orbit,

$$C_{nl} = \langle V_{ls}(r) \rangle_{nl},$$

the first-order shifts are

$$\Delta E_{ls} \left(l + \frac{1}{2}\right) = C_{nl} \frac{\hbar^2 l}{2},$$

and

$$\Delta E_{ls} \left(l - \frac{1}{2}\right) = -C_{nl} \frac{\hbar^2(l+1)}{2}.$$

The separation between the two spin-orbit partners is consequently

$$E_{l+1/2} - E_{l-1/2} = C_{nl} \frac{\hbar^2}{2} (2l+1).$$

Thus the magnitude of the splitting increases approximately with $2l+1$.

In nuclei the effective spin-orbit coupling has the sign for which the aligned state

$$j = l + \frac{1}{2}$$

lies below the anti-aligned state

$$j = l - \frac{1}{2}.$$

With the convention used above, this corresponds to $C_{nl} < 0$. Each l -multiplet, originally containing

$$2(2l+1)$$

states including spin, is divided into two levels:

$$j = l + \frac{1}{2}, \quad 2j+1 = 2l+2,$$

and

$$j = l - \frac{1}{2}, \quad 2j+1 = 2l.$$

Their total number of magnetic substates is conserved:

$$(2l+2) + (2l) = 2(2l+1).$$

For $l = 0$, only $j = 1/2$ exists and

$$\mathbf{l} \cdot \mathbf{s} = 0,$$

so there is no spin-orbit splitting of an s -orbit.

CONCLUSION:

The basic shell-model assumption is that every proton or neutron moves approximately independently in a common, spherically symmetric mean field generated by all the other nucleons, with the Pauli principle enforced and residual interactions treated as corrections.

The ideal three-dimensional harmonic oscillator has energies

$$E_N^{(0)} = \left(N + \frac{3}{2}\right) \hbar\omega, \quad N = 2n_r + l,$$

so different l -orbits belonging to the same N -shell are accidentally degenerate. The centrifugal barrier

$$\frac{\hbar^2 l(l+1)}{2mr^2}$$

makes different l -states experience different effective radial potentials. In an exact harmonic oscillator its

effect is cancelled in the total energy, but in a realistic non-oscillator nuclear mean field this cancellation is lost, producing an l -dependent splitting.

The spin-orbit interaction further divides every $l \neq 0$ orbit into the partners

$$j = l + \frac{1}{2} \quad \text{and} \quad j = l - \frac{1}{2},$$

with

$$\langle \mathbf{l} \cdot \mathbf{s} \rangle = \begin{cases} \frac{\hbar^2 l}{2}, & j = l + \frac{1}{2}, \\ -\frac{\hbar^2(l+1)}{2}, & j = l - \frac{1}{2}. \end{cases}$$

For the nuclear sign of the coupling, the aligned $j = l + \frac{1}{2}$ level is lower, and the splitting has magnitude

$$|E_{l+1/2} - E_{l-1/2}| = |C_n| \frac{\hbar^2}{2} (2l+1).$$

The combined l -dependent central-field effect and strong spin-orbit splitting destroy the large oscillator degeneracy and generate the observed nuclear single-particle shell structure.



UPSC PHYSICS OPTIONAL for CSE & IFoS

ABHI PHYSICS

A SYSTEMATIC ARCHIVE OF SOLVED PHYSICS OPTIONAL PYQs

Detailed, source-tagged solutions arranged for focused study,
conceptual reconstruction and reliable examination reference.

Nuclear and Particle Physics

41 YEARS | 1986–2026

10 SOLVED QUESTIONS | CSE + IFoS

Published by Abhi Physics

For Civil Services and Indian Forest Service Examinations

www.abhi-physics.com