



UPSC PHYSICS OPTIONAL for CSE & IFoS

ABHI PHYSICS

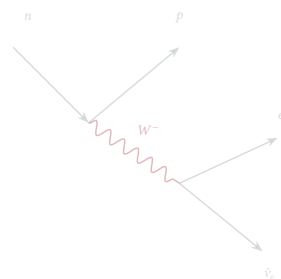
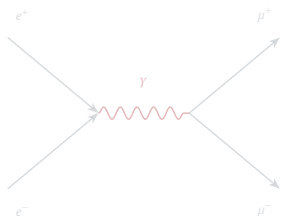
$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$E_n = -\frac{13.6}{n^2} \text{ eV}$$

PREVIOUS YEAR QUESTIONS



41 YEARS QUESTION BANK | 1986-2026

CSE + IFoS

SUBJECT

Mechanics

10 SOLVED QUESTIONS • CSE + IFoS

$$n \rightarrow p + e^- + \bar{\nu}_e$$

$$E^2 = p^2 c^2 + m^2 c^4$$

$$\partial_\mu j^\mu = 0$$

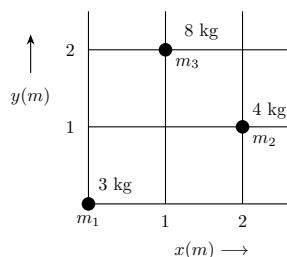
SUBJECT-WISE • TOPIC-WISE • SOURCE-TAGGED

For Civil Services and Indian Forest Service Examinations

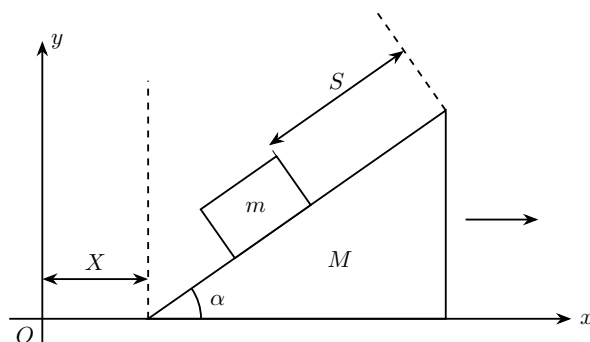
Sample Preview extbar 10 Solved Questions

Question Index

No.	Question	Marks	Year	Exam	Page
1	(i) Define cyclic coordinates and find their connections with the symmetries of the system.; (ii) A system with two degrees of freedom is described by the Lagrangian $L = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 - \frac{\alpha}{q_1}$, α is constant. Find the cyclic coordinates, conserved quantities and symmetries for this system, if any.	10+5=15	2025	IFoS	3
2	A particle of mass m kg having an initial velocity V_0 is subjected to a retarding force proportional to its instantaneous velocity. Obtain the expression for the velocity and position of the particle as a function of time.	10	2024	CSE	6
3	The Lagrangian of a system is given by $L(\vec{r}, \vec{v}) = \frac{1}{2}m(\vec{v} + \vec{a})^2 + \vec{b} \cdot \vec{v} - \vec{c} \cdot \vec{r}$ where, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are constant vectors. Construct the Hamiltonian of the system and find the canonical equations of motion.	8	2024	IFoS	8
4	Construct the Lagrangian of a simple pendulum whose string is replaced by a spring with spring constant k and rest length l_0 oscillating in a vertical $x - y$ plane. Find the equations of motion.	10	2024	IFoS	10
5	A force $\vec{F}$ is given by $\vec{F} = x^2y\hat{x} + zy^2\hat{y} + xz^2\hat{z}$. Determine whether or not the force is conservative.	10	2023	CSE	13
6	What are the coordinates of the centre of mass of the system of masses shown in the figure?	8	2023	IFoS	14



- | | | | | | |
|---|--|----|------|------|----|
| 7 | Consider a particle of mass m moving in a potential field of the form $V(\vec{r}) = V_0(x^2 + y^2)$, where V_0 is a constant. What are the conserved physical quantities for the particle ? | 8 | 2022 | IFoS | 15 |
| 8 | A small block of mass m slides without friction down a wedge-shaped block of mass M and opening angle α as depicted below. | 15 | 2022 | IFoS | 17 |



The wedge-shaped block itself slides along the horizontal frictionless floor along the +ve x -direction. Find the horizontal acceleration $\ddot{X} = d^2X/dt^2$ of the wedge-shaped block by the Lagrangian method and using the displacements X and S of the blocks as generalized coordinates.

- | | | | | | |
|---|---|----|------|-----|----|
| 9 | A body of mass m at rest splits into two masses m_1 and m_2 by an explosion. After the split the bodies move with a total kinetic energy T in opposite direction. Show that their relative speed is $\sqrt{\frac{2Tm}{m_1m_2}}$. | 15 | 2021 | CSE | 18 |
|---|---|----|------|-----|----|

No.	Question	Marks	Year	Exam	Page
10	During take-off, an aircraft accelerates horizontally in a straight line at a rate A . A small bob of mass m is suspended on a string attached to the roof of the cabin, and a hydrogen balloon (total mass m) is tethered to the floor by a string. For each, determine the tension in the string and the equilibrium angle θ between string and vertical. Draw a neat diagram to explain your answer.	10	2021	IFoS	19

IFoS • 2025 • 10+5=15 MARKS

Q1. (i) Define cyclic coordinates and find their connections with the symmetries of the system.; (ii) A system with two degrees of freedom is described by the Lagrangian $L = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2q_1^2\dot{q}_2^2 - \frac{\alpha}{q_1}$, α is constant. Find the cyclic coordinates, conserved quantities and symmetries for this system, if any.

INTRODUCTION:

The problem asks for the relation between cyclic coordinates, continuous symmetries, and conservation laws, and then for its application to the given two-degree-of-freedom system.

Let the generalized coordinates be q_i , with generalized velocities $\dot{q}_i$, and let

$$L = L(q_i, \dot{q}_i, t).$$

A coordinate q_k is called cyclic, ignorable, or redundant when the Lagrangian has no explicit dependence on that coordinate:

$$\frac{\partial L}{\partial q_k} = 0.$$

The Lagrangian may nevertheless depend on the corresponding velocity $\dot{q}_k$. The conjugate generalized momentum is

$$p_k = \frac{\partial L}{\partial \dot{q}_k}.$$

The relevant governing principles are the Euler–Lagrange equations and Noether’s theorem. A cyclic coordinate generates a continuous translation in configuration space,

$$q_k \longrightarrow q_k + \varepsilon, \quad \delta q_k = \varepsilon,$$

where ε is a constant infinitesimal parameter. Since the Lagrangian is independent of q_k , this transformation leaves the Lagrangian and hence the action invariant. Noether’s theorem then identifies p_k as the conserved charge.

For the given system, m_1, m_2 , and α are taken to be constants, $m_1, m_2 > 0$, and the motion is restricted to a regular domain with $q_1 \neq 0$. The positive q_2 -direction is defined as the direction of increasing q_2 ; consequently, the sign of p_2 specifies the sense of motion along that coordinate. No particular physical dimensions are assigned to q_1 and q_2 , although if q_1 is a radial coordinate and q_2 an angle, the q_2 -translation is an ordinary rotation.

SOLUTION:

(i) The Euler–Lagrange equation for q_k is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0.$$

If q_k is cyclic, then

$$\frac{\partial L}{\partial q_k} = 0,$$

and hence

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) = 0.$$

Therefore,

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \text{constant.}$$

The connection with symmetry follows directly. Under the infinitesimal transformation

$$\delta q_i = \varepsilon \delta_{ik}, \quad \delta \dot{q}_i = 0, \quad \delta t = 0,$$

the change in the Lagrangian is

$$\delta L = \sum_i \frac{\partial L}{\partial q_i} \delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i = \varepsilon \frac{\partial L}{\partial q_k} = 0.$$

Thus the action

$$S = \int L dt$$

is invariant under translations along the q_k -direction.

More generally, on using the Euler–Lagrange equations, the variation of L under a continuous coordinate transformation can be written as

$$\delta L = \frac{d}{dt} \left(\sum_i p_i \delta q_i \right).$$

For the cyclic-coordinate transformation $\delta q_k = \varepsilon$, this becomes

$$0 = \delta L = \varepsilon \frac{dp_k}{dt},$$

which again gives

$$\frac{dp_k}{dt} = 0.$$

Hence the chain of implications is

$$\begin{aligned} \frac{\partial L}{\partial q_k} = 0 &\iff q_k \rightarrow q_k + \varepsilon \text{ is a continuous symmetry} \\ &\implies p_k = \text{constant.} \end{aligned}$$

The equivalence here refers to an ordinary constant translation of the coordinate for which the Lagrangian itself is invariant. More general transformations may leave the action invariant up to a total time derivative and lead to the generalized form of Noether’s theorem.

If the Lagrangian has no explicit time dependence,

$$\frac{\partial L}{\partial t} = 0,$$

the system is also invariant under time translations

$$t \longrightarrow t + \varepsilon.$$

The corresponding conserved Noether charge is the energy

$$E = \sum_i \dot{q}_i p_i - L.$$

Time is not one of the configuration coordinates q_i , so this is an additional symmetry rather than an additional cyclic coordinate among the given q_i .

(ii) The Lagrangian is

$$L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 q_1^2 \dot{q}_2^2 - \frac{\alpha}{q_1}.$$

Its explicit coordinate derivatives are

$$\frac{\partial L}{\partial q_1} = m_2 q_1 \dot{q}_2^2 + \frac{\alpha}{q_1^2}, \quad \frac{\partial L}{\partial q_2} = 0.$$

Therefore q_2 is cyclic, whereas q_1 is not cyclic for the stated Lagrangian.

The conjugate momenta are

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = m_1 \dot{q}_1,$$

and

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} = m_2 q_1^2 \dot{q}_2.$$

The Euler–Lagrange equation for q_2 gives

$$\frac{d}{dt} (m_2 q_1^2 \dot{q}_2) = 0.$$

Thus

$$p_2 = m_2 q_1^2 \dot{q}_2 = J = \text{constant},$$

where J denotes the conserved generalized momentum associated with the q_2 -symmetry.

Equivalently,

$$\dot{q}_2 = \frac{J}{m_2 q_1^2}.$$

Thus, when $J > 0$, the motion is in the direction of increasing q_2 ; when $J < 0$, it is in the direction of decreasing q_2 ; and when $J = 0$, q_2 remains constant as long as $q_1 \neq 0$.

The corresponding finite continuous transformation is

$$q'_1 = q_1, \quad q'_2 = q_2 + \varepsilon, \quad t' = t.$$

Since

$$\dot{q}'_1 = \dot{q}_1, \quad \dot{q}'_2 = \dot{q}_2,$$

one has

$$L(q'_1, q'_2, \dot{q}'_1, \dot{q}'_2) = L(q_1, q_2, \dot{q}_1, \dot{q}_2).$$

Therefore the continuous symmetry group contains translations along q_2 . If q_2 is an angular coordinate, this is rotational symmetry, $q_2 \mapsto q_2 + \varepsilon$, and J has the interpretation of the corresponding angular momentum. Without that geometrical interpretation, J should simply be called the conserved generalized momentum conjugate to q_2 .

For completeness, the equation of motion for the non-cyclic coordinate q_1 is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_1} \right) - \frac{\partial L}{\partial q_1} = 0,$$

or

$$m_1 \ddot{q}_1 - m_2 q_1 \dot{q}_2^2 - \frac{\alpha}{q_1^2} = 0.$$

Hence

$$m_1 \ddot{q}_1 = m_2 q_1 \dot{q}_2^2 + \frac{\alpha}{q_1^2}.$$

Substitution of the conserved quantity

$$\dot{q}_2 = \frac{J}{m_2 q_1^2}$$

gives

$$m_1 \ddot{q}_1 = \frac{J^2}{m_2 q_1^3} + \frac{\alpha}{q_1^2}.$$

This explicitly shows that no analogous constant p_1 follows, because the generalized force conjugate to q_1 is generally nonzero.

The Lagrangian also contains no explicit time dependence:

$$\frac{\partial L}{\partial t} = 0.$$

It is therefore invariant under time translations,

$$t \longrightarrow t + \varepsilon,$$

and the energy is conserved. Using

$$E = \dot{q}_1 p_1 + \dot{q}_2 p_2 - L,$$

one obtains

$$\begin{aligned} E &= m_1 \dot{q}_1^2 + m_2 q_1^2 \dot{q}_2^2 - \left(\frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 q_1^2 \dot{q}_2^2 - \frac{\alpha}{q_1} \right) \\ &= \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 q_1^2 \dot{q}_2^2 + \frac{\alpha}{q_1}. \end{aligned}$$

Therefore,

$$E = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2q_1^2\dot{q}_2^2 + \frac{\alpha}{q_1} = \text{constant}.$$

Eliminating $\dot{q}_2$ in favour of J gives the one-dimensional energy form

$$E = \frac{1}{2}m_1\dot{q}_1^2 + \frac{J^2}{2m_2q_1^2} + \frac{\alpha}{q_1}.$$

Thus an effective potential for the q_1 -motion is

$$V_{\text{eff}}(q_1) = \frac{J^2}{2m_2q_1^2} + \frac{\alpha}{q_1}.$$

Indeed,

$$-m_1^{-1} \frac{dV_{\text{eff}}}{dq_1} = \frac{1}{m_1} \left(\frac{J^2}{m_2q_1^3} + \frac{\alpha}{q_1^2} \right),$$

which agrees with the directly derived equation for $\ddot{q}_1$.

The conservation of energy can also be checked explicitly:

$$\begin{aligned} \frac{dE}{dt} &= m_1\dot{q}_1\ddot{q}_1 + m_2q_1\dot{q}_1\dot{q}_2^2 + m_2q_1^2\dot{q}_2\ddot{q}_2 - \frac{\alpha}{q_1^2}\dot{q}_1 \\ &= \dot{q}_1 \left(m_1\ddot{q}_1 - m_2q_1\dot{q}_2^2 - \frac{\alpha}{q_1^2} \right) + \dot{q}_2 \frac{d}{dt} (m_2q_1^2\dot{q}_2) \\ &= 0, \end{aligned}$$

where both Euler–Lagrange equations have been used.

The dimensions are consistent because every term in L and E has the dimensions of energy:

$$[m_1\dot{q}_1^2] = [m_2q_1^2\dot{q}_2^2] = \left[\frac{\alpha}{q_1} \right] = [E].$$

Consequently,

$$[\alpha] = [E][q_1], \quad [J] = [p_2] = \left[\frac{E}{\dot{q}_2} \right].$$

If q_2 is dimensionless, as for an angle measured in radians, then

$$[J] = [E][T],$$

the dimensions of angular momentum. In SI units this is $\text{J s} = \text{kg m}^2 \text{s}^{-1}$, while E is measured in joules.

As a limiting check, if $J = 0$, then

$$\dot{q}_2 = 0, \quad E = \frac{1}{2}m_1\dot{q}_1^2 + \frac{\alpha}{q_1},$$

so the system reduces consistently to one-dimensional q_1 -motion. If q_1 is constant during a motion, conservation of J requires $\dot{q}_2$ to be constant. For $\alpha > 0$, $q_1 > 0$, and $J \neq 0$, both terms on the right-hand side of

$$m_1\ddot{q}_1 = \frac{J^2}{m_2q_1^3} + \frac{\alpha}{q_1^2}$$

are positive, confirming that the acceleration is toward increasing q_1 under the adopted sign convention.

There is also a discrete reflection invariance

$$q_2 \longrightarrow -q_2, \quad \dot{q}_2 \longrightarrow -\dot{q}_2,$$

provided the coordinate domain admits this reflection. It changes J to $-J$ and therefore reverses the sense of motion. Because it is a discrete rather than a continuously parameterized symmetry, it does not by itself produce an additional Noether conserved quantity. Similarly, the velocity-quadratic autonomous Lagrangian is invariant under time reversal, which reverses both generalized momenta but introduces no further independent continuous Noether charge.

CONCLUSION:

A cyclic coordinate is one that is absent explicitly from the Lagrangian,

$$\frac{\partial L}{\partial q_k} = 0.$$

It represents invariance under translations in the corresponding configuration-space direction,

$$q_k \rightarrow q_k + \varepsilon,$$

and its conjugate momentum

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

is conserved.

For the given system, the only cyclic coordinate among q_1 and q_2 is

$$q_2.$$

Its continuous symmetry and conserved generalized momentum are

$$q_2 \rightarrow q_2 + \varepsilon, \quad J = p_2 = m_2q_1^2\dot{q}_2 = \text{constant}.$$

The sign of J gives the direction of motion: $J > 0$ corresponds to increasing q_2 , $J < 0$ to decreasing q_2 , and $J = 0$ to fixed q_2 . If q_2 is an angle, this symmetry is rotational invariance and J is angular momentum with SI unit J s .

Since L is autonomous, the system also has time-translation symmetry,

$$t \rightarrow t + \varepsilon,$$

with conserved energy

$$\begin{aligned} E &= \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2q_1^2\dot{q}_2^2 + \frac{\alpha}{q_1} \\ &= \frac{1}{2}m_1\dot{q}_1^2 + \frac{J^2}{2m_2q_1^2} + \frac{\alpha}{q_1} = \text{constant}. \end{aligned}$$

The energy has SI unit joule, and dimensional consistency requires

$$[\alpha] = [E][q_1].$$

The results are valid for constant m_1, m_2, α , with $m_1, m_2 > 0$, on any regular portion of the motion for which $q_1 \neq 0$. The physical inference is that the independence of the dynamics from the absolute value of q_2 makes only changes in q_2 physically relevant and enforces conservation of its conjugate generalized momentum, while invariance under a shift of the time origin enforces conservation of total energy.

CSE • 2024 • 10 MARKS

Q2. A particle of mass m kg having an initial velocity V_0 is subjected to a retarding force proportional to its instantaneous velocity. Obtain the expression for the velocity and position of the particle as a function of time.

INTRODUCTION:

The problem requires the velocity and position of a particle moving under linear resistive damping. Let the motion be one-dimensional along the x -axis, and choose the positive x -direction to be the direction of the initial velocity. Thus,

$$v(0) = V_0 > 0, \quad x(0) = x_0.$$

Since the force is retarding, it acts opposite to the instantaneous velocity. If $k > 0$ is the constant of proportionality, the force law is

$$F_x = -kv,$$

where k is assumed constant and the particle is subject to no other force along the direction of motion. Newton's second law then gives the governing equation

$$m \frac{dv}{dt} = -kv.$$

The minus sign is fixed by the stated coordinate convention: while $v > 0$, the force and acceleration are

in the negative x -direction. The magnitude of the retarding force is kv , so

$$[k] = \frac{[F]}{[v]} = \frac{\text{kg m s}^{-2}}{\text{m s}^{-1}} = \text{kg s}^{-1}.$$

Consequently, m/k has the dimensions of time. Define the relaxation time

$$\tau = \frac{m}{k}.$$

SOLUTION:

Using Newton's second law and the linear retarding-force law,

$$m \frac{dv}{dt} = -kv,$$

or

$$\frac{1}{v} dv = -\frac{k}{m} dt.$$

Integrating from the initial state $(t, v) = (0, V_0)$ to the state $(t, v(t))$,

$$\int_{V_0}^{v(t)} \frac{dv'}{v'} = -\frac{k}{m} \int_0^t dt',$$

which gives

$$\ln\left(\frac{v(t)}{V_0}\right) = -\frac{k}{m}t.$$

Exponentiation therefore yields

$$v(t) = V_0 \exp\left(-\frac{k}{m}t\right) = V_0 e^{-t/\tau}.$$

The acceleration is

$$a(t) = \frac{dv}{dt} = -\frac{kV_0}{m} \exp\left(-\frac{k}{m}t\right).$$

For $V_0 > 0$, this is negative at every finite time, confirming that the force is indeed retarding. Moreover,

$$ma(t) = -kV_0 \exp\left(-\frac{k}{m}t\right) = -kv(t),$$

so the obtained velocity satisfies the original equation of motion.

To determine the position, use

$$\frac{dx}{dt} = v(t) = V_0 \exp\left(-\frac{k}{m}t\right).$$

Integrating from $x(0) = x_0$ to $x(t)$,

$$x(t) - x_0 = \int_0^t V_0 \exp\left(-\frac{k}{m}t'\right) dt'.$$

Since

$$\int \exp\left(-\frac{k}{m}t'\right) dt' = -\frac{m}{k} \exp\left(-\frac{k}{m}t'\right),$$

one obtains

$$x(t) - x_0 = V_0 \left[-\frac{m}{k} \exp\left(-\frac{k}{m}t'\right) \right]_0^t$$

and hence

$$x(t) = x_0 + \frac{mV_0}{k} \left(1 - \exp\left(-\frac{k}{m}t\right) \right) = x_0 + V_0\tau \left(1 - e^{-t/\tau} \right).$$

If the origin is chosen at the initial position, so that $x_0 = 0$, this reduces to

$$x(t) = \frac{mV_0}{k} \left(1 - e^{-kt/m} \right).$$

Differentiating the position expression verifies the kinematic relation:

$$\frac{dx}{dt} = \frac{mV_0}{k} \left(\frac{k}{m} \right) e^{-kt/m} = V_0 e^{-kt/m} = v(t).$$

The dimensions are consistent. The exponential argument is dimensionless because

$$\left[\frac{kt}{m} \right] = \frac{(\text{kg s}^{-1})(\text{s})}{\text{kg}} = 1,$$

and the displacement scale has dimensions

$$\left[\frac{mV_0}{k} \right] = \frac{\text{kg}(\text{m s}^{-1})}{\text{kg s}^{-1}} = \text{m}.$$

For short times $t \ll \tau$, using

$$e^{-t/\tau} = 1 - \frac{t}{\tau} + \frac{t^2}{2\tau^2} + \dots,$$

the velocity and position become

$$v(t) \simeq V_0 \left(1 - \frac{t}{\tau} \right) = V_0 - \frac{kV_0}{m}t,$$

and

$$x(t) \simeq x_0 + V_0t - \frac{kV_0}{2m}t^2.$$

Thus the initial acceleration is

$$a(0) = -\frac{kV_0}{m},$$

as required directly by $ma(0) = -kV_0$.

The zero-damping limit also recovers uniform motion. As $k \rightarrow 0$,

$$v(t) \rightarrow V_0,$$

while

$$\lim_{k \rightarrow 0} \frac{mV_0}{k} \left(1 - e^{-kt/m} \right) = V_0t,$$

so that

$$x(t) \rightarrow x_0 + V_0t.$$

At long times,

$$\lim_{t \rightarrow \infty} v(t) = 0, \quad \lim_{t \rightarrow \infty} x(t) = x_0 + \frac{mV_0}{k}.$$

Therefore, although the particle approaches zero speed only asymptotically, it travels only the finite total distance

$$s_\infty = \frac{mV_0}{k}$$

from its initial position.

As an energy check, the kinetic energy is

$$K(t) = \frac{1}{2}mv^2(t) = \frac{1}{2}mV_0^2 e^{-2kt/m}.$$

Its rate of change is

$$\frac{dK}{dt} = mv \frac{dv}{dt} = v(-kv) = -kv^2 \leq 0.$$

This equals the power delivered by the retarding force,

$$P = F_x v = (-kv)v = -kv^2,$$

confirming that mechanical energy is continuously dissipated and that the sign convention is physically consistent.

CONCLUSION:

For the linear retarding force

$$F_x = -kv, \quad k > 0,$$

with $x(0) = x_0$ and $v(0) = V_0$, the velocity and position are

$$v(t) = V_0 e^{-kt/m} \quad \text{in m s}^{-1},$$

and

$$x(t) = x_0 + \frac{mV_0}{k} \left(1 - e^{-kt/m} \right) \quad \text{in m}.$$

If $x_0 = 0$,

$$x(t) = \frac{mV_0}{k} \left(1 - e^{-kt/m} \right).$$

The characteristic decay time is

$$\tau = \frac{m}{k} \quad \text{in s}.$$

For $V_0 > 0$, the acceleration and retarding force point in the negative x -direction, while the velocity remains in the positive x -direction and decreases exponentially. The motion approaches rest asymptotically, and the total distance travelled is finite:

$$s_\infty = \frac{mV_0}{k} \quad \text{in m}.$$

These results hold when the mass and damping coefficient are constant, the resistive force is exactly proportional to and opposite the instantaneous velocity, and no other force acts along the direction of motion.

IFoS • 2024 • 8 MARKS

Q3. The Lagrangian of a system is given by $L(\vec{r}, \vec{v}) = \frac{1}{2}m(\vec{v} + \vec{a})^2 + \vec{b} \cdot \vec{v} - \vec{c} \cdot \vec{r}$ where, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are constant vectors. Construct the Hamiltonian of the system and find the canonical equations of motion.

INTRODUCTION:

The generalized coordinate is the position vector $\vec{r}$, and the corresponding generalized velocity is

$$\vec{v} = \dot{\vec{r}}.$$

The Hamiltonian is obtained by the Legendre transformation

$$H(\vec{r}, \vec{p}) = \vec{p} \cdot \vec{v} - L(\vec{r}, \vec{v}),$$

where the canonical momentum is

$$\vec{p} = \frac{\partial L}{\partial \vec{v}}.$$

Since $\vec{a}$, $\vec{b}$, and $\vec{c}$ are constant vectors and $m \neq 0$, the relation between $\vec{p}$ and $\vec{v}$ is invertible. Thus the Lagrangian is regular and a Hamiltonian formulation exists. All vectors and derivatives are taken in a fixed inertial Cartesian frame. The canonical equations are

$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}}, \quad \dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}}.$$

SOLUTION:

Differentiating the Lagrangian with respect to $\vec{v}$, component by component, gives

$$p_i = \frac{\partial L}{\partial v_i} = m(v_i + a_i) + b_i.$$

Hence, in vector form,

$$\vec{p} = m(\vec{v} + \vec{a}) + \vec{b}.$$

The Hessian with respect to the velocity components is

$$\frac{\partial^2 L}{\partial v_i \partial v_j} = m\delta_{ij}, \quad \det\left(\frac{\partial^2 L}{\partial v_i \partial v_j}\right) = m^3 \neq 0,$$

which explicitly verifies that the velocity can be expressed uniquely in terms of the canonical momentum.

Solving the momentum relation for $\vec{v}$,

$$m(\vec{v} + \vec{a}) = \vec{p} - \vec{b},$$

so that

$$\vec{v} + \vec{a} = \frac{\vec{p} - \vec{b}}{m}, \quad \vec{v} = \frac{\vec{p} - \vec{b}}{m} - \vec{a}.$$

The Hamiltonian is

$$H = \vec{p} \cdot \vec{v} - L.$$

Substituting the Lagrangian,

$$H = \vec{p} \cdot \vec{v} - \frac{1}{2}m(\vec{v} + \vec{a})^2 - \vec{b} \cdot \vec{v} + \vec{c} \cdot \vec{r}.$$

Combining the terms linear in $\vec{v}$,

$$H = (\vec{p} - \vec{b}) \cdot \vec{v} - \frac{1}{2}m(\vec{v} + \vec{a})^2 + \vec{c} \cdot \vec{r}.$$

Using

$$\vec{p} - \vec{b} = m(\vec{v} + \vec{a}), \quad \vec{v} = \frac{\vec{p} - \vec{b}}{m} - \vec{a},$$

the first term becomes

$$(\vec{p} - \vec{b}) \cdot \vec{v} = (\vec{p} - \vec{b}) \cdot \left(\frac{\vec{p} - \vec{b}}{m} - \vec{a} \right) = \frac{(\vec{p} - \vec{b})^2}{m} - \vec{a} \cdot (\vec{p} - \vec{b}),$$

while

$$\frac{1}{2}m(\vec{v} + \vec{a})^2 = \frac{1}{2}m \left(\frac{\vec{p} - \vec{b}}{m} \right)^2 = \frac{(\vec{p} - \vec{b})^2}{2m}.$$

Therefore,

$$H(\vec{r}, \vec{p}) = \frac{(\vec{p} - \vec{b})^2}{2m} - \vec{a} \cdot (\vec{p} - \vec{b}) + \vec{c} \cdot \vec{r}.$$

Equivalently, on completing the square,

$$H(\vec{r}, \vec{p}) = \frac{(\vec{p} - \vec{b} - m\vec{a})^2}{2m} + \vec{c} \cdot \vec{r} - \frac{1}{2}m\vec{a}^2.$$

The two forms are identical; the final constant term in the second form does not affect Hamilton's equations.

For the first canonical equation,

$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}}.$$

Since

$$\frac{\partial}{\partial \vec{p}} \left[\frac{(\vec{p} - \vec{b})^2}{2m} \right] = \frac{\vec{p} - \vec{b}}{m}, \quad \frac{\partial}{\partial \vec{p}} [-\vec{a} \cdot (\vec{p} - \vec{b})] = -\vec{a},$$

one obtains

$$\dot{\vec{r}} = \frac{\vec{p} - \vec{b}}{m} - \vec{a}.$$

This agrees with the previously inverted momentum–velocity relation and therefore provides an internal consistency check.

For the second canonical equation,

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}}.$$

Only the term $\vec{c} \cdot \vec{r}$ depends on $\vec{r}$, and hence

$$\frac{\partial H}{\partial \vec{r}} = \vec{c}.$$

Therefore,

$$\dot{\vec{p}} = -\vec{c}.$$

The physical equation of motion follows by differentiating the first canonical equation. Since $\vec{a}$ and $\vec{b}$ are constant,

$$\ddot{\vec{r}} = \frac{\dot{\vec{p}}}{m}.$$

Using $\dot{\vec{p}} = -\vec{c}$,

$$m\ddot{\vec{r}} = -\vec{c}.$$

Thus the system describes a particle subjected to the constant force

$$\vec{F} = -\vec{c}.$$

For initial data

$$\vec{r}(0) = \vec{r}_0, \quad \vec{p}(0) = \vec{p}_0,$$

the canonical equations can also be integrated explicitly:

$$\vec{p}(t) = \vec{p}_0 - \vec{c}t,$$

and

$$\dot{\vec{r}}(t) = \frac{\vec{p}_0 - \vec{b}}{m} - \vec{a} - \frac{\vec{c}}{m}t.$$

Therefore,

$$\vec{r}(t) = \vec{r}_0 + \left(\frac{\vec{p}_0 - \vec{b}}{m} - \vec{a} \right)t - \frac{\vec{c}}{2m}t^2.$$

The dimensions are consistent:

$$[\vec{p}] = [\vec{b}] = MLT^{-1}$$

$$[\vec{a}] = LT^{-1}$$

$$[\vec{c}] = MLT^{-2}.$$

Consequently,

$$\left[\frac{(\vec{p} - \vec{b})^2}{2m} \right] = [\vec{a} \cdot (\vec{p} - \vec{b})] = [\vec{c} \cdot \vec{r}] = ML^2T^{-2},$$

so every term in H has the dimension of energy.

Because the Lagrangian and Hamiltonian have no explicit time dependence,

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0.$$

The Hamiltonian is therefore conserved along the motion. Indeed, using Hamilton's equations,

$$\frac{dH}{dt} = \frac{\partial H}{\partial \vec{r}} \cdot \dot{\vec{r}} + \frac{\partial H}{\partial \vec{p}} \cdot \dot{\vec{p}} = \vec{c} \cdot \dot{\vec{r}} + \dot{\vec{r}} \cdot (-\vec{c}) = 0.$$

As a limiting check, if

$$\vec{a} = \vec{b} = \vec{0},$$

then

$$L = \frac{1}{2}m\vec{v}^2 - \vec{c} \cdot \vec{r}, \quad H = \frac{\vec{p}^2}{2m} + \vec{c} \cdot \vec{r},$$

and Hamilton's equations reduce to

$$\dot{\vec{r}} = \frac{\vec{p}}{m}, \quad \dot{\vec{p}} = -\vec{c},$$

which are the standard equations for a particle in the linear potential

$$V(\vec{r}) = \vec{c} \cdot \vec{r}.$$

The role of the constant vectors $\vec{a}$ and $\vec{b}$ can also be seen by expanding the Lagrangian:

$$L = \frac{1}{2}m\vec{v}^2 + (m\vec{a} + \vec{b}) \cdot \vec{v} + \frac{1}{2}m\vec{a}^2 - \vec{c} \cdot \vec{r}.$$

Since

$$(m\vec{a} + \vec{b}) \cdot \vec{v} = \frac{d}{dt} [(m\vec{a} + \vec{b}) \cdot \vec{r}],$$

the velocity-linear term is a total time derivative, while $\frac{1}{2}m\vec{a}^2$ is a constant. These terms alter the definition of the canonical momentum and shift the Hamiltonian by canonical-momentum and constant terms, but they do not change the physical equation $m\ddot{\vec{r}} = -\vec{c}$.

CONCLUSION:

The canonical momentum is

$$\vec{p} = m(\dot{\vec{r}} + \vec{a}) + \vec{b}.$$

The Hamiltonian, expressed entirely in terms of the canonical variables $(\vec{r}, \vec{p})$, is

$$H(\vec{r}, \vec{p}) = \frac{(\vec{p} - \vec{b})^2}{2m} - \vec{a} \cdot (\vec{p} - \vec{b}) + \vec{c} \cdot \vec{r}$$

or, equivalently,

$$H(\vec{r}, \vec{p}) = \frac{(\vec{p} - \vec{b} - m\vec{a})^2}{2m} + \vec{c} \cdot \vec{r} - \frac{1}{2}m\vec{a}^2.$$

The canonical equations of motion are

$$\dot{\vec{r}} = \frac{\vec{p} - \vec{b}}{m} - \vec{a}, \quad \dot{\vec{p}} = -\vec{c}.$$

They imply

$$m\ddot{\vec{r}} = -\vec{c},$$

so the particle has the constant acceleration

$$\ddot{\vec{r}} = -\frac{\vec{c}}{m},$$

directed opposite to $\vec{c}$. This result is valid for constant $\vec{a}$, $\vec{b}$, and $\vec{c}$, with $m \neq 0$. The vectors $\vec{a}$ and $\vec{b}$ shift the relation between canonical momentum and mechanical velocity but do not change the physical force, which is $\vec{F} = -\vec{c}$.

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Q4. Construct the Lagrangian of a simple pendulum whose string is replaced by a spring with spring constant k and rest length l_0 oscillating in a vertical $x-y$ plane. Find the equations of motion.

INTRODUCTION:

The system is an elastic pendulum consisting of a point mass m attached to the lower end of a massless Hookean spring. The upper end of the spring is fixed at a pivot O , and the motion is restricted to a vertical plane. The problem requires the Lagrangian and the coupled nonlinear equations governing the radial extension and angular motion.

Choose an inertial Cartesian frame with origin at O , the x -axis horizontal towards the right, and the y -axis vertically upward. Let $r(t) > 0$ be the instantaneous length of the spring, and let $\theta(t)$ be the angle measured from the vertically downward direction towards the positive x -direction. Thus,

$$x = r \sin \theta, \quad y = -r \cos \theta.$$

The generalized coordinates are

$$q_1 = r, \quad q_2 = \theta.$$

The following assumptions are made:

1. The bob is a point mass m , while the spring is massless.
2. The spring remains straight and obeys Hooke's law throughout the motion.
3. The motion is planar, and the pivot is fixed in an inertial frame.
4. Damping and air resistance are absent, so all forces are conservative.
5. The spring length remains positive, $r > 0$.

With the outward radial unit vector directed from the pivot towards the mass, Hooke's law gives

$$\mathbf{F}_s = -k(r - l_0)\hat{\mathbf{e}}_r.$$

Thus, when $r > l_0$, the stretched spring pulls the mass inward.

The governing principle is Hamilton's principle,

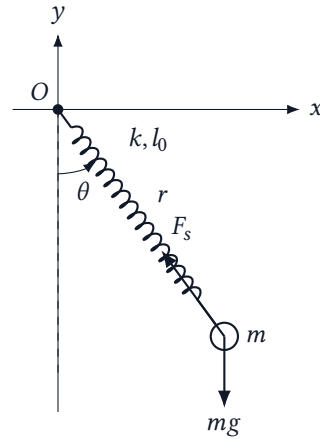
$$\delta \int_{t_1}^{t_2} L dt = 0, \quad L = T - V,$$

which leads to the Euler-Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0.$$

SOLUTION:

The coordinate and sign conventions are represented by the following diagram.



From

$$x = r \sin \theta, \quad y = -r \cos \theta,$$

the velocity components are

$$\dot{x} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta,$$

and

$$\dot{y} = -\dot{r} \cos \theta + r \dot{\theta} \sin \theta.$$

Therefore,

$$\begin{aligned} \dot{x}^2 + \dot{y}^2 &= (\dot{r} \sin \theta + r \dot{\theta} \cos \theta)^2 + (-\dot{r} \cos \theta + r \dot{\theta} \sin \theta)^2 \\ &= \dot{r}^2 (\sin^2 \theta + \cos^2 \theta) + r^2 \dot{\theta}^2 (\sin^2 \theta + \cos^2 \theta) \\ &\quad + 2r\dot{r}\dot{\theta} \sin \theta \cos \theta - 2r\dot{r}\dot{\theta} \sin \theta \cos \theta \\ &= \dot{r}^2 + r^2 \dot{\theta}^2. \end{aligned}$$

Hence the kinetic energy is

$$T = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2).$$

Taking the gravitational potential energy to be zero at the level of the pivot,

$$V_g = mgy = -mgr \cos \theta.$$

The elastic potential energy of the spring is

$$V_s = \frac{k}{2}(r - l_0)^2.$$

Thus,

$$V = V_s + V_g = \frac{k}{2}(r - l_0)^2 - mgr \cos \theta.$$

The Lagrangian is consequently

$$L(r, \theta, \dot{r}, \dot{\theta}) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{2}(r - l_0)^2 + mgr \cos \theta.$$

For the radial coordinate r ,

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = m\ddot{r},$$

and

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - k(r - l_0) + mg \cos \theta.$$

The Euler–Lagrange equation for r is therefore

$$m\ddot{r} - \{mr\dot{\theta}^2 - k(r - l_0) + mg \cos \theta\} = 0.$$

Hence,

$$m(\ddot{r} - r\dot{\theta}^2) = mg \cos \theta - k(r - l_0).$$

Equivalently,

$$\ddot{r} - r\dot{\theta}^2 = g \cos \theta - \frac{k}{m}(r - l_0).$$

This equation has the expected radial-force interpretation. The outward radial component of the acceleration is

$$a_r = \ddot{r} - r\dot{\theta}^2.$$

The component of gravity along the outward radial direction is

$$mg \cos \theta,$$

whereas the spring force is

$$-k(r - l_0).$$

Thus the radial equation is also obtained directly from Newton's second law along $\hat{\mathbf{e}}_r$.

For the angular coordinate θ ,

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}.$$

Therefore,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}.$$

Also,

$$\frac{\partial L}{\partial \theta} = -mgr \sin \theta.$$

The angular Euler–Lagrange equation becomes

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr \sin \theta = 0.$$

For $m > 0$ and $r > 0$, division by mr gives

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + g \sin \theta = 0.$$

Equivalently,

$$\ddot{\theta} + 2\frac{\dot{r}}{r}\dot{\theta} + \frac{g}{r} \sin \theta = 0.$$

The term

$$2\frac{\dot{r}}{r}\dot{\theta}$$

expresses the coupling between radial and angular motion due to the changing distance of the mass from the pivot.

The angular equation can also be checked by taking moments about O . Since the spring force passes through O , it exerts no torque about the pivot. The angular momentum about O is

$$\mathcal{L}_O = mr^2\dot{\theta},$$

while the gravitational torque in the positive- θ convention is

$$\tau_O = -mgr \sin \theta.$$

Thus,

$$\frac{d}{dt} (mr^2\dot{\theta}) = -mgr \sin \theta,$$

which reproduces the angular equation.

For the static downward equilibrium,

$$\theta_{\text{eq}} = 0, \quad \dot{r} = \dot{\theta} = \ddot{r} = \ddot{\theta} = 0.$$

The radial equation then gives

$$0 = mg - k(r_{\text{eq}} - l_0),$$

so that

$$r_{\text{eq}} = l_0 + \frac{mg}{k}.$$

Therefore, gravity produces the physically expected extension

$$r_{\text{eq}} - l_0 = \frac{mg}{k} > 0.$$

At the inverted position $\theta = \pi$,

$$mg \cos \pi = -mg,$$

so gravity has an inward radial component, consistently with the adopted convention.

The dimensional consistency of the radial equation follows from

$$[\ddot{r}] = [r\dot{\theta}^2] = [g] = \text{m s}^{-2},$$

and

$$\left[\frac{k}{m}(r - l_0) \right] = \frac{Nm^{-1}}{kg} m = \frac{kg s^{-2}}{kg} m = m s^{-2}.$$

In the angular equation,

$$[\ddot{\theta}] = \left[\frac{\dot{r}\dot{\theta}}{r} \right] = \left[\frac{g}{r} \right] = s^{-2},$$

because the angle in radians is dimensionless.

Since the Lagrangian has no explicit time dependence, the corresponding energy function

$$E = \dot{r} \frac{\partial L}{\partial \dot{r}} + \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} - L$$

is conserved. Substitution gives

$$E = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{k}{2} (r - l_0)^2 - mgr \cos \theta.$$

This conservation law may be verified directly without assuming the result. Differentiating E gives

$$\begin{aligned} \frac{dE}{dt} &= m\dot{r}\ddot{r} + mr\dot{\theta}^2 + mr^2\ddot{\theta} + k(r - l_0)\dot{r} \\ &\quad - mgr\dot{\theta} \cos \theta + mgr\dot{\theta} \sin \theta. \end{aligned}$$

Regrouping the terms according to the two equations of motion,

$$\begin{aligned} \frac{dE}{dt} &= \dot{r} [m\ddot{r} - mr\dot{\theta}^2 + k(r - l_0) - mg \cos \theta] \\ &\quad + \dot{\theta} [mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr \sin \theta]. \end{aligned}$$

Each square bracket vanishes by the corresponding Euler-Lagrange equation. Therefore,

$$\frac{dE}{dt} = 0,$$

confirming conservation of total mechanical energy.

As a limiting check, suppose the spring is sufficiently stiff that radial departures from equilibrium are negligible. Then

$$r \simeq r_{eq} = \text{constant}, \quad \dot{r} \simeq 0.$$

The angular equation reduces to

$$\ddot{\theta} + \frac{g}{r_{eq}} \sin \theta = 0,$$

which is the nonlinear equation of an ordinary simple pendulum of length r_{eq} . For small angular displacements,

$$\sin \theta \simeq \theta,$$

and hence

$$\ddot{\theta} + \frac{g}{r_{eq}} \theta = 0.$$

Thus,

$$\omega_{\theta} \simeq \sqrt{\frac{g}{r_{eq}}},$$

as expected for a simple pendulum.

For small oscillations about the stable downward equilibrium, write

$$r = r_{eq} + \rho, \quad |\rho| \ll r_{eq}, \quad |\theta| \ll 1.$$

To first order in the small quantities,

$$\sin \theta \simeq \theta, \quad \cos \theta \simeq 1,$$

while

$$r\dot{\theta}^2, \quad \dot{\rho}\dot{\theta}, \quad \rho\theta$$

are of second order and are neglected. Using

$$k(r_{eq} - l_0) = mg,$$

the radial equation becomes

$$m\ddot{\rho} = -k\rho,$$

or

$$\ddot{\rho} + \frac{k}{m} \rho = 0.$$

The angular equation becomes

$$\ddot{\theta} + \frac{g}{r_{eq}} \theta = 0.$$

Thus the radial and angular motions decouple to linear order, with frequencies

$$\omega_r = \sqrt{\frac{k}{m}}, \quad \omega_{\theta} = \sqrt{\frac{g}{r_{eq}}} = \sqrt{\frac{g}{l_0 + mg/k}}.$$

CONCLUSION:

For a point mass m suspended from a massless Hookean spring of spring constant k and rest length l_0 , with r measured outward from the pivot and θ measured from the vertically downward direction towards positive x , the Lagrangian is

$$L = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{k}{2} (r - l_0)^2 + mgr \cos \theta.$$

The coupled nonlinear equations of motion are

$$m(\ddot{r} - r\dot{\theta}^2) = mg \cos \theta - k(r - l_0)$$

in the outward radial direction, and

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + g \sin \theta = 0$$

in the direction of increasing θ .

Here r and l_0 are measured in m, m in kg, k in N m^{-1} , θ in radians, and g in m s^{-2} . The stable downward equilibrium is

$$\theta_{\text{eq}} = 0, \quad r_{\text{eq}} = l_0 + \frac{mg}{k},$$

so the equilibrium extension is mg/k in the downward direction.

The conserved mechanical energy is

$$E = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{k}{2} (r - l_0)^2 - mgr \cos \theta = \text{constant}.$$

The equations are valid for planar motion with $r > 0$, a fixed pivot, a point bob, a massless spring obeying Hooke's law, and negligible dissipative forces. In the stiff-spring limit the angular equation reduces to the ordinary simple-pendulum equation. Near the stable equilibrium, the radial and angular small-oscillation frequencies are

$$\omega_r = \sqrt{\frac{k}{m}}, \quad \omega_\theta = \sqrt{\frac{g}{l_0 + mg/k}}.$$

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Q5. A force $\vec{F}$ is given by $\vec{F} = x^2 y \hat{x} + zy^2 \hat{y} + xz^2 \hat{z}$. Determine whether or not the force is conservative.

INTRODUCTION:

The demand is to determine whether the given force field can be expressed as the gradient of a single-valued scalar potential. Write

$$\begin{aligned} \vec{F} &= F_x \hat{x} + F_y \hat{y} + F_z \hat{z} \\ F_x &= x^2 y \\ F_y &= zy^2 \\ F_z &= xz^2. \end{aligned}$$

The field is polynomial and hence continuously differentiable throughout the simply connected domain $\mathbb{R}^3$. Therefore, a necessary and sufficient condition for $\vec{F}$ to be conservative on this domain is

$$\nabla \times \vec{F} = \vec{0}$$

at every point. Equivalently, the relevant mixed partial derivatives of the components must agree. The standard right-handed Cartesian coordinate system is used.

SOLUTION:

The curl of the force field is

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & zy^2 & xz^2 \end{vmatrix}.$$

Expanding the determinant,

$$\nabla \times \vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z}.$$

The required derivatives are

$$\frac{\partial F_z}{\partial y} = \frac{\partial(xz^2)}{\partial y} = 0, \quad \frac{\partial F_y}{\partial z} = \frac{\partial(zy^2)}{\partial z} = y^2,$$

$$\frac{\partial F_x}{\partial z} = \frac{\partial(x^2 y)}{\partial z} = 0, \quad \frac{\partial F_z}{\partial x} = \frac{\partial(xz^2)}{\partial x} = z^2,$$

and

$$\frac{\partial F_y}{\partial x} = \frac{\partial(zy^2)}{\partial x} = 0, \quad \frac{\partial F_x}{\partial y} = \frac{\partial(x^2 y)}{\partial y} = x^2.$$

Hence

$$\nabla \times \vec{F} = -y^2 \hat{x} - z^2 \hat{y} - x^2 \hat{z}.$$

This curl is not identically zero. It vanishes only at the isolated point

$$x = y = z = 0,$$

which is insufficient: conservativeness requires the curl to vanish throughout an open domain.

The conclusion can also be checked directly through the equality of mixed derivatives that would follow from a potential V . If

$$\vec{F} = -\nabla V,$$

then, for example,

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}.$$

But here

$$\frac{\partial F_x}{\partial y} = x^2, \quad \frac{\partial F_y}{\partial x} = 0,$$

which are unequal for $x \neq 0$. Thus no scalar potential can generate the given field throughout $\mathbb{R}^3$.

As an independent circulation check, consider a rectangle in the plane $z = 0$, with

$$a \leq x \leq b, \quad c \leq y \leq d,$$

oriented counterclockwise as viewed from $+\hat{z}$. By Stokes' theorem,

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{z} dA.$$

Since

$$(\nabla \times \vec{F}) \cdot \hat{z} = -x^2,$$

one obtains

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = - \int_c^d \int_a^b x^2 dx dy = - \frac{(d-c)(b^3 - a^3)}{3},$$

which is generally nonzero. This confirms that the work depends on the path and that the work around a closed path need not vanish.

Dimensionally, each component of $\nabla \times \vec{F}$ has the dimensions of force per unit length, as expected after taking one spatial derivative of a force field. For a loop whose dimensions shrink to zero, the circulation also tends to zero with the enclosed area and the local curl; this limiting behavior is consistent with Stokes' theorem but does not make the field conservative on any finite open region where the curl is nonzero.

CONCLUSION:

The given force field is

not conservative

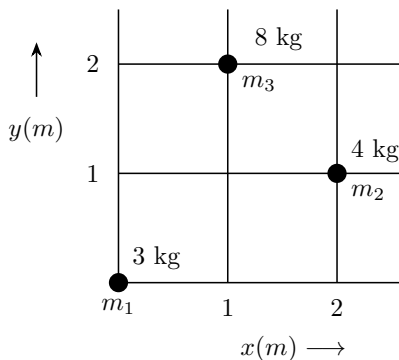
on $\mathbb{R}^3$, because

$$\nabla \times \vec{F} = -y^2 \hat{x} - z^2 \hat{y} - x^2 \hat{z} \neq \vec{0}$$

in general. Consequently, no single-valued scalar potential exists for this force over $\mathbb{R}^3$, the work done is path-dependent, and the circulation around a closed curve can be nonzero.

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Q6. What are the coordinates of the centre of mass of the system of masses shown in the figure?



INTRODUCTION:

The system consists of three point masses in the xy -plane. Taking the origin and the positive x - and y -directions exactly as indicated in the figure, their masses and position vectors are

$$m_1 = 3 \text{ kg}, \quad \vec{r}_1 = (0 \text{ m}, 0 \text{ m}),$$

$$m_2 = 4 \text{ kg}, \quad \vec{r}_2 = (2 \text{ m}, 1 \text{ m}),$$

and

$$m_3 = 8 \text{ kg}, \quad \vec{r}_3 = (1 \text{ m}, 2 \text{ m}).$$

For discrete point masses, the centre-of-mass position vector is the mass-weighted mean

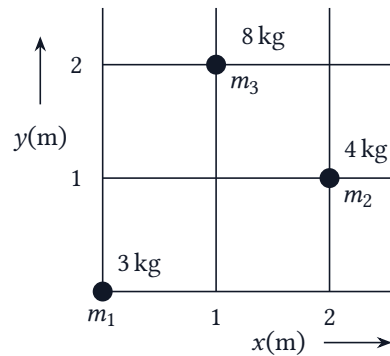
$$\vec{R}_{\text{CM}} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}.$$

Hence its Cartesian coordinates are

$$x_{\text{CM}} = \frac{\sum_i m_i x_i}{\sum_i m_i}, \quad y_{\text{CM}} = \frac{\sum_i m_i y_i}{\sum_i m_i}.$$

SOLUTION:

The configuration represented by the given figure is



The total mass is

$$M = m_1 + m_2 + m_3 = 3 \text{ kg} + 4 \text{ kg} + 8 \text{ kg} = 15 \text{ kg}.$$

The x -coordinate of the centre of mass is

$$\begin{aligned} x_{\text{CM}} &= \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{M} \\ &= \frac{(3 \text{ kg})(0 \text{ m}) + (4 \text{ kg})(2 \text{ m}) + (8 \text{ kg})(1 \text{ m})}{15 \text{ kg}} \\ &= \frac{16 \text{ kg m}}{15 \text{ kg}} = \frac{16}{15} \text{ m} \approx 1.07 \text{ m}. \end{aligned}$$

Similarly, the y -coordinate is

$$\begin{aligned} y_{\text{CM}} &= \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{M} \\ &= \frac{(3 \text{ kg})(0 \text{ m}) + (4 \text{ kg})(1 \text{ m}) + (8 \text{ kg})(2 \text{ m})}{15 \text{ kg}} \\ &= \frac{20 \text{ kg m}}{15 \text{ kg}} = \frac{4}{3} \text{ m} \approx 1.33 \text{ m}. \end{aligned}$$

Thus

$$\vec{R}_{\text{CM}} = \left(\frac{16}{15} \hat{i} + \frac{4}{3} \hat{j} \right) \text{ m}.$$

Both coordinates lie between the minimum and maximum particle coordinates, $0 \leq x_{\text{CM}} \leq 2 \text{ m}$ and $0 \leq$

$y_{CM} \leq 2 \text{ m}$, as required for a positive-mass weighted mean.

CONCLUSION:

With respect to the origin and axes shown in the figure, the coordinates of the centre of mass are

$$(x_{CM}, y_{CM}) = \left(\frac{16}{15}, \frac{4}{3} \right) \text{ m} \approx (1.07, 1.33) \text{ m}.$$

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Q7. Consider a particle of mass m moving in a potential field of the form $V(\vec{r}) = V_0(x^2 + y^2)$, where V_0 is a constant. What are the conserved physical quantities for the particle ?

INTRODUCTION:

The demand is to identify the constants of motion by examining the continuous symmetries of the potential. We use a fixed Cartesian inertial frame with

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}, \quad \rho = \sqrt{x^2 + y^2},$$

and take positive rotations about the z -axis according to the right-hand rule. The potential can then be written as

$$V(\vec{r}) = V_0\rho^2.$$

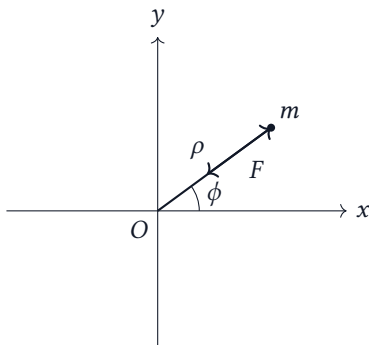
It is independent of time t , the coordinate z , and the azimuthal angle ϕ . By Noether's theorem, these invariances imply conservation of total mechanical energy, linear momentum along the z -axis, and angular momentum about the z -axis, respectively. The motion is assumed to be governed by the non-relativistic Lagrangian

$$\mathcal{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V_0(x^2 + y^2),$$

with m and V_0 constant and with no non-conservative or explicitly time-dependent forces.

SOLUTION:

The geometry and the force directions in the transverse plane are represented by:



Since

$$\nabla V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z} = 2V_0x\hat{x} + 2V_0y\hat{y},$$

the force is

$$\vec{F} = -\nabla V = -2V_0x\hat{x} - 2V_0y\hat{y}, \quad F_z = 0.$$

For $V_0 > 0$, the transverse force points toward the z -axis; for $V_0 < 0$, it points away from that axis. The conservation laws themselves hold for either sign of V_0 .

1. Conservation of total mechanical energy

The potential has no explicit time dependence:

$$\frac{\partial V}{\partial t} = 0.$$

The total mechanical energy is therefore

$$E = T + V = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V_0(x^2 + y^2).$$

To verify its conservation directly,

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + m\dot{y}\ddot{y} + m\dot{z}\ddot{z} + 2V_0x\dot{x} + 2V_0y\dot{y}.$$

The equations of motion are

$$m\ddot{x} = -2V_0x, \quad m\ddot{y} = -2V_0y, \quad m\ddot{z} = 0.$$

Substitution gives

$$\frac{dE}{dt} = \dot{x}(-2V_0x) + \dot{y}(-2V_0y) + 0 + 2V_0x\dot{x} + 2V_0y\dot{y} = 0.$$

Hence

$$E = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + V_0(x^2 + y^2) = \text{constant}.$$

2. Conservation of the z -component of linear momentum

Since V is independent of z , the coordinate z is cyclic:

$$\frac{\partial V}{\partial z} = 0.$$

Consequently,

$$\dot{p}_z = F_z = -\frac{\partial V}{\partial z} = 0,$$

and therefore

$$p_z = m\dot{z} = \text{constant}.$$

Thus the motion parallel to the z -axis is uniform:

$$z(t) = z_0 + \frac{p_z}{m}t.$$

The full linear momentum is not generally conserved, because

$$\dot{p}_x = -2V_0x, \quad \dot{p}_y = -2V_0y$$

are not generally zero.

3. Conservation of angular momentum about the z-axis

The potential depends on x and y only through $x^2 + y^2 = \rho^2$. It is therefore invariant under rotations

$$\begin{aligned}x &\longrightarrow x \cos \alpha - y \sin \alpha \\y &\longrightarrow x \sin \alpha + y \cos \alpha\end{aligned}$$

about the z -axis. The corresponding conserved quantity is

$$L_z = (\vec{r} \times \vec{p})_z = x p_y - y p_x = m(x\dot{y} - y\dot{x}).$$

Direct differentiation yields

$$\frac{dL_z}{dt} = x\dot{p}_y - y\dot{p}_x = x(-2V_0\dot{y}) - y(-2V_0\dot{x}) = 0.$$

Equivalently, the z -component of the torque is

$$\tau_z = (\vec{r} \times \vec{F})_z = xF_y - yF_x = 0.$$

Thus

$$L_z = x p_y - y p_x = m(x\dot{y} - y\dot{x}) = \text{constant.}$$

The entire angular-momentum vector is not generally conserved. Indeed,

$$\vec{\tau} = \vec{r} \times \vec{F} = 2V_0 z (y \hat{x} - x \hat{y}),$$

so τ_x and τ_y need not vanish, whereas $\tau_z = 0$.

4. Additional constants arising from separability

Because the x , y , and z motions are dynamically separable, the energies assigned to the individual Cartesian motions are separately constant:

$$\begin{aligned}E_x &= \frac{p_x^2}{2m} + V_0 x^2 \\E_y &= \frac{p_y^2}{2m} + V_0 y^2 \\E_z &= \frac{p_z^2}{2m}.\end{aligned}$$

For example,

$$\frac{dE_x}{dt} = \frac{p_x \dot{p}_x}{m} + 2V_0 x \dot{x} = \dot{x}(-2V_0 x) + 2V_0 x \dot{x} = 0,$$

and similarly $dE_y/dt = 0$. Also, E_z is constant because p_z is constant. These quantities satisfy

$$E = E_x + E_y + E_z,$$

so they are not all independent of the total energy. The symmetry-generated quantities normally requested are E , p_z , and L_z .

For $V_0 > 0$, defining

$$\omega^2 = \frac{2V_0}{m},$$

the transverse equations become

$$\ddot{x} + \omega^2 x = 0, \quad \ddot{y} + \omega^2 y = 0.$$

Thus the transverse motion is an isotropic harmonic oscillation, while the z -motion is free. This provides a plausibility check: isotropy in the x - y plane requires conservation of L_z , and absence of any z -directed force requires conservation of p_z .

Dimensionally,

$$[V_0] = \frac{[V]}{[x^2]} = \frac{\text{J}}{\text{m}^2} = \text{kg s}^{-2},$$

so $V_0 x^2$ has the dimension of energy and $2V_0 x$ has the dimension of force. Moreover,

$$\left[\frac{2V_0}{m} \right] = \text{s}^{-2},$$

as required for ω^2 . In the limiting case $V_0 \rightarrow 0$, the force vanishes and all three components of linear momentum and all three components of angular momentum become conserved, consistently enlarging the set of conservation laws for a free particle.

CONCLUSION:

For a particle moving in

$$V(\vec{r}) = V_0(x^2 + y^2),$$

the principal conserved physical quantities are

$$E = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + V_0(x^2 + y^2), \quad p_z = m\dot{z}, \quad L_z = x p_y - y p_x.$$

They follow, respectively, from invariance under time translations, translations along the z -axis, and rotations about the z -axis. The direction associated with the conserved linear momentum is the z -direction, and the conserved angular-momentum component is the component along the z -axis. The separate quantities

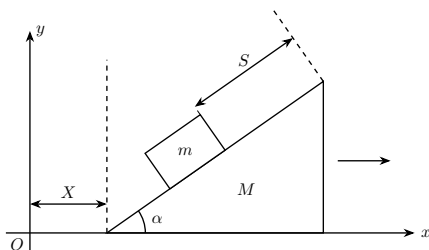
$$E_x = \frac{p_x^2}{2m} + V_0 x^2, \quad E_y = \frac{p_y^2}{2m} + V_0 y^2, \quad E_z = \frac{p_z^2}{2m}$$

are also constants of motion because the Cartesian equations are separable, with $E = E_x + E_y + E_z$. In general, neither the full linear-momentum vector nor the full angular-momentum vector is conserved. For $V_0 > 0$, the particle executes bounded isotropic harmonic motion transverse to the z -axis with $\omega = \sqrt{2V_0/m}$, together with uniform motion parallel to the z -axis;

for $V_0 < 0$, the transverse motion is unstable, but the stated conservation laws remain valid.

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Q8. A small block of mass m slides without friction down a wedge-shaped block of mass M and opening angle α as depicted below.



The wedge-shaped block itself slides along the horizontal frictionless floor along the +ve x -direction. Find the horizontal acceleration $\ddot{X} = d^2X/dt^2$ of the wedge-shaped block by the Lagrangian method and using the displacements X and S of the blocks as generalized coordinates.

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INTRODUCTION:

The required quantity is the horizontal acceleration of the wedge in the inertial frame fixed to the floor. Let the positive x -axis point horizontally to the right and the positive y -axis vertically upward. The coordinate X denotes the horizontal displacement of a reference point fixed to the wedge. The angle α is the angle made by the inclined face with the horizontal.

Let S be the displacement of the small block measured along the inclined face from a fixed upper point of the wedge, with S increasing as the block slides downward and to the left. Thus, increasing S has horizontal and vertical components $-S \cos \alpha$ and $-S \sin \alpha$, respectively.

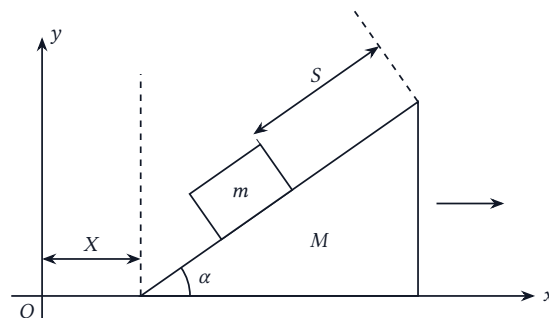
The floor and the contact between the two blocks are frictionless, both blocks are treated as rigid, and the small block remains in contact with the incline.

Hence the only generalized coordinates are X and S . The governing principle is Lagrange's equation,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \quad L = T - V.$$

SOLUTION:

The configuration and the coordinate conventions are shown below.



If (x_m, y_m) are the coordinates of the small block, then, apart from constants fixed by the choice of reference point,

$$x_m = X - S \cos \alpha + \text{constant}$$

$$y_m = -S \sin \alpha + \text{constant}.$$

Therefore,

$$\dot{x}_m = \dot{X} - \dot{S} \cos \alpha, \quad \dot{y}_m = -\dot{S} \sin \alpha.$$

The kinetic energy of the wedge is

$$T_M = \frac{1}{2} M \dot{X}^2,$$

while that of the small block is

$$T_m = \frac{1}{2} m (\dot{x}_m^2 + \dot{y}_m^2)$$

$$= \frac{1}{2} m [(\dot{X} - \dot{S} \cos \alpha)^2 + \dot{S}^2 \sin^2 \alpha]$$

$$= \frac{1}{2} m \dot{X}^2 - m \dot{X} \dot{S} \cos \alpha + \frac{1}{2} m \dot{S}^2.$$

Hence the total kinetic energy is

$$T = \frac{1}{2} (M + m) \dot{X}^2 - m \dot{X} \dot{S} \cos \alpha + \frac{1}{2} m \dot{S}^2.$$

Taking the gravitational potential energy to vanish at any convenient reference level,

$$V = mgy_m = -mgS \sin \alpha + \text{constant}.$$

The irrelevant additive constant may be omitted, giving the Lagrangian

$$L = \frac{1}{2} (M + m) \dot{X}^2 - m \dot{X} \dot{S} \cos \alpha + \frac{1}{2} m \dot{S}^2 + mgS \sin \alpha.$$

Since X does not occur explicitly in L , it is a cyclic coordinate. Its Lagrange equation gives

$$\frac{d}{dt} [(M + m) \dot{X} - m \dot{S} \cos \alpha] = 0,$$

or

$$(M + m)\ddot{X} - m\ddot{S} \cos \alpha = 0. \quad (1)$$

This equation is also the statement of conservation of the total horizontal momentum, as expected because there is no external horizontal force.

For the coordinate S ,

$$\frac{\partial L}{\partial \dot{S}} = m\dot{S} - m\dot{X} \cos \alpha, \quad \frac{\partial L}{\partial S} = mg \sin \alpha.$$

Thus,

$$m\ddot{S} - m\ddot{X} \cos \alpha - mg \sin \alpha = 0,$$

and hence

$$\ddot{S} = g \sin \alpha + \ddot{X} \cos \alpha. \quad (2)$$

Substituting equation (2) into equation (1),

$$(M + m)\ddot{X} = m \cos \alpha (g \sin \alpha + \ddot{X} \cos \alpha),$$

$$(M + m - m \cos^2 \alpha) \ddot{X} = mg \sin \alpha \cos \alpha.$$

Using $1 - \cos^2 \alpha = \sin^2 \alpha$,

$$M + m - m \cos^2 \alpha = M + m \sin^2 \alpha.$$

Therefore,

$$\ddot{X} = \frac{mg \sin \alpha \cos \alpha}{M + m \sin^2 \alpha}.$$

For $0 < \alpha < \pi/2$, both $\sin \alpha$ and $\cos \alpha$ are positive, so $\ddot{X} > 0$. The wedge consequently accelerates toward the right, opposite to the leftward horizontal component of the small block's motion. Also, $\ddot{X} \rightarrow 0$ for either $\alpha \rightarrow 0$ or $\alpha \rightarrow \pi/2$, as expected: in the first case gravity has no component along the surface, while in the second case the block's motion has no horizontal component.

CONCLUSION:

The horizontal acceleration of the wedge-shaped block is

$$\ddot{X} = \frac{mg \sin \alpha \cos \alpha}{M + m \sin^2 \alpha}$$

in units of m s^{-2} . For the configuration shown, with $0 < \alpha < \pi/2$ and S increasing downward along the incline, this acceleration is along the positive x -direction, i.e. horizontally to the right. The result applies while both contacts remain frictionless and the small block remains in contact with the inclined face.

CSE • 2021 • 15 MARKS

Q9. A body of mass m at rest splits into two masses m_1 and m_2 by an explosion. After the

split the bodies move with a total kinetic energy T in opposite direction. Show that their relative speed is $\sqrt{\frac{2Tm}{m_1 m_2}}$.

INTRODUCTION:

The demand is to determine the relative speed of the two fragments using conservation of linear momentum and the given total kinetic energy. We work in the initial rest frame of the body and assume non-relativistic motion, negligible external impulse during the explosion, and conservation of mass in the form

$$m = m_1 + m_2.$$

Choose the x -axis along the motion of mass m_1 . After the explosion, let

$$\vec{v}_1 = v_1 \hat{x}, \quad \vec{v}_2 = -v_2 \hat{x},$$

where $v_1, v_2 > 0$. Thus, the fragments move in opposite directions. Their relative velocity is

$$\vec{v}_{12} = \vec{v}_1 - \vec{v}_2 = (v_1 + v_2) \hat{x},$$

so their relative speed is $v_{\text{rel}} = v_1 + v_2$.

SOLUTION:

Since the original body is at rest, its initial momentum is zero. The explosion is an internal process and the external impulse during it is negligible; hence linear momentum is conserved:

$$m_1 \vec{v}_1 + m_2 \vec{v}_2 = 0.$$

Using the stated sign convention,

$$m_1 v_1 - m_2 v_2 = 0,$$

and therefore

$$m_1 v_1 = m_2 v_2.$$

Let the common magnitude of the fragment momenta be p . Then

$$p = m_1 v_1 = m_2 v_2, \quad v_1 = \frac{p}{m_1}, \quad v_2 = \frac{p}{m_2}.$$

The total kinetic energy after the explosion is

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2.$$

Substituting the velocities in terms of p ,

$$T = \frac{1}{2} m_1 \left(\frac{p}{m_1} \right)^2 + \frac{1}{2} m_2 \left(\frac{p}{m_2} \right)^2,$$

so

$$T = \frac{p^2}{2} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) = \frac{p^2}{2} \frac{m_1 + m_2}{m_1 m_2}.$$

Since $m_1 + m_2 = m$,

$$T = \frac{p^2 m}{2 m_1 m_2},$$

which gives

$$p^2 = \frac{2Tm_1m_2}{m}.$$

Because the velocities are oppositely directed, the relative speed is the sum of their individual speeds:

$$v_{\text{rel}} = v_1 + v_2.$$

Hence

$$v_{\text{rel}} = \frac{p}{m_1} + \frac{p}{m_2} = p \left(\frac{1}{m_1} + \frac{1}{m_2} \right) = p \frac{m}{m_1m_2}.$$

Squaring and substituting for p^2 ,

$$v_{\text{rel}}^2 = p^2 \frac{m^2}{m_1^2m_2^2} = \left(\frac{2Tm_1m_2}{m} \right) \frac{m^2}{m_1^2m_2^2} = \frac{2Tm}{m_1m_2}.$$

Therefore,

$$v_{\text{rel}} = \sqrt{\frac{2Tm}{m_1m_2}}.$$

Equivalently, introducing the reduced mass

$$\mu = \frac{m_1m_2}{m_1 + m_2} = \frac{m_1m_2}{m},$$

the total kinetic energy in the zero-momentum frame is

$$T = \frac{1}{2} \mu v_{\text{rel}}^2,$$

which immediately gives the same result.

The dimensional check is

$$\left[\frac{Tm}{m_1m_2} \right] = \frac{(ML^2T^{-2})M}{M^2} = L^2T^{-2},$$

so its square root has the dimension of speed. Also, as $T \rightarrow 0$, the relative speed tends to zero, as physically required. For $m_1 = m_2 = m/2$, momentum conservation gives equal and opposite speeds, and the formula reduces consistently to

$$v_{\text{rel}} = \sqrt{\frac{8T}{m}}.$$

CONCLUSION:

The fragments move in opposite directions along their common line of motion, with momenta equal in magnitude and opposite in direction. Under the assumptions of negligible external impulse, non-relativistic motion, and $m = m_1 + m_2$, their relative speed is

$$v_{\text{rel}} = \sqrt{\frac{2Tm}{m_1m_2}}.$$

The relative velocity of m_1 with respect to m_2 is directed along the motion of m_1 , while that of m_2 with

respect to m_1 has the same magnitude and the opposite direction.

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Q10. During take-off, an aircraft accelerates horizontally in a straight line at a rate A . A small bob of mass m is suspended on a string attached to the roof of the cabin, and a hydrogen balloon (total mass m) is tethered to the floor by a string. For each, determine the tension in the string and the equilibrium angle θ between string and vertical. Draw a neat diagram to explain your answer.

INTRODUCTION:

Take the aircraft acceleration to be constant and directed along the positive horizontal x -axis:

$$\mathbf{A} = A\hat{\mathbf{x}},$$

with the positive y -axis vertically upward. The calculation will be made in the non-inertial frame fixed to the aircraft. In this frame, every object of mass M experiences the pseudo-force

$$\mathbf{F}_p = -MA\hat{\mathbf{x}}.$$

Consequently, gravity and the pseudo-force together correspond to the effective gravitational acceleration

$$\mathbf{g}_{\text{eff}} = -A\hat{\mathbf{x}} - g\hat{\mathbf{y}}, \quad g_{\text{eff}} = \sqrt{g^2 + A^2}.$$

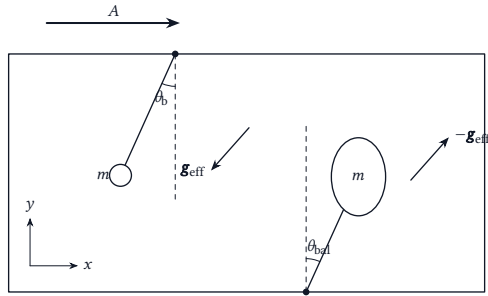
The strings are assumed light, inextensible, and taut. The bob and the balloon are in equilibrium relative to the aircraft. The bob is treated as a point mass. For the balloon, let V be its displaced volume and let ρ be the local density of the cabin air, taken approximately uniform over the dimensions of the balloon. The total balloon mass m includes the hydrogen, envelope, and all attached parts. The cabin air is assumed to have reached mechanical equilibrium in the accelerating frame, so that its pressure gradient is determined by gravity and the horizontal pseudo-force.

For the bob, θ_b is measured from the vertically downward line through the roof attachment. For the balloon, θ_{bal} is measured from the vertically upward line through the floor attachment. Angles toward the positive x -direction are described as forward, while angles toward the negative x -direction are described as backward.

SOLUTION:

1. The geometry and sign convention are shown below. The aircraft accelerates to the right.

The bob is displaced backward, whereas a sufficiently buoyant balloon is displaced forward.



The arrow A points to the right and denotes the actual acceleration of the aircraft. The effective gravity $\mathbf{g}_{\text{eff}}$ points downward and backward. The pressure-produced buoyant force on the balloon will be shown to point opposite to $\mathbf{g}_{\text{eff}}$, namely upward and forward.

2. For the bob, the real forces are its weight

$$\mathbf{W} = -mg\hat{\mathbf{y}}$$

and the tension exerted by the roof string. Since the bob lies behind the roof attachment, the tension on the bob points upward and forward:

$$\mathbf{T}_b = T_b \sin \theta_b \hat{\mathbf{x}} + T_b \cos \theta_b \hat{\mathbf{y}}.$$

The pseudo-force in the aircraft frame is

$$\mathbf{F}_p = -mA\hat{\mathbf{x}}.$$

Equilibrium relative to the aircraft requires

$$\mathbf{T}_b + \mathbf{W} + \mathbf{F}_p = \mathbf{0}.$$

Thus

$$T_b \sin \theta_b \hat{\mathbf{x}} + T_b \cos \theta_b \hat{\mathbf{y}} - mA\hat{\mathbf{x}} - mg\hat{\mathbf{y}} = \mathbf{0}.$$

Equating the horizontal and vertical components gives

$$T_b \sin \theta_b = mA, \quad T_b \cos \theta_b = mg.$$

Dividing the first equation by the second,

$$\tan \theta_b = \frac{A}{g},$$

and hence

$$\theta_b = \tan^{-1}\left(\frac{A}{g}\right).$$

Squaring and adding the component equations,

$$T_b^2 (\sin^2 \theta_b + \cos^2 \theta_b) = m^2 A^2 + m^2 g^2,$$

so that

$$T_b = m\sqrt{g^2 + A^2}.$$

Equivalently, the string aligns the bob with the effective gravity:

$$\mathbf{g}_{\text{eff}} = -A\hat{\mathbf{x}} - g\hat{\mathbf{y}}.$$

Its horizontal component is backward, confirming that the bob must trail behind the accelerating aircraft.

3. For the balloon, ordinary vertical buoyancy alone is insufficient because the accelerating cabin air also has a horizontal pressure gradient. To derive the pressure gradient, consider a small cabin-air element of density ρ at rest in the aircraft frame. The forces per unit volume are the pressure force

$$-\nabla p,$$

gravity

$$-\rho g\hat{\mathbf{y}},$$

and the pseudo-force

$$-\rho A\hat{\mathbf{x}}.$$

Static equilibrium of the air therefore gives

$$-\nabla p - \rho A\hat{\mathbf{x}} - \rho g\hat{\mathbf{y}} = \mathbf{0},$$

or

$$\boxed{\nabla p = -\rho A\hat{\mathbf{x}} - \rho g\hat{\mathbf{y}} = \rho \mathbf{g}_{\text{eff}}}.$$

Thus pressure decreases both upward and forward.

The net pressure force on a body occupying volume V is

$$\mathbf{F}_{\text{pressure}} = -\oint_{\partial V} p \mathbf{n} dS,$$

where $\mathbf{n}$ is the outward unit normal. Applying the divergence theorem component by component, and taking the pressure gradient as approximately constant over the balloon,

$$\mathbf{F}_{\text{pressure}} = -\int_V \nabla p dV = -V\nabla p.$$

Using the pressure-gradient equation,

$$\mathbf{F}_{\text{pressure}} = \rho V A\hat{\mathbf{x}} + \rho V g\hat{\mathbf{y}}.$$

This is the generalized buoyant force in the accelerating cabin. It is directed upward and forward, opposite to $\mathbf{g}_{\text{eff}}$, and has magnitude

$$F_{\text{pressure}} = \rho V \sqrt{g^2 + A^2}.$$

The weight and pseudo-force acting on the total balloon mass m are

$$\mathbf{W} + \mathbf{F}_p = -mg\hat{\mathbf{y}} - mA\hat{\mathbf{x}}.$$

Therefore, before including the tether tension, the resultant force on the balloon is

$$\begin{aligned}\mathbf{F}_{\text{free}} &= \mathbf{F}_{\text{pressure}} + \mathbf{W} + \mathbf{F}_p \\ &= (\rho V - m)A\hat{\mathbf{x}} + (\rho V - m)g\hat{\mathbf{y}} \\ &= (\rho V - m)(A\hat{\mathbf{x}} + g\hat{\mathbf{y}}).\end{aligned}$$

For a balloon that can be held by a taut floor tether,

$$\rho V > m.$$

Under this condition, $\mathbf{F}_{\text{free}}$ points upward and forward. The balloon consequently displaces forward until the floor tether pulls it downward and backward.

The tension exerted on the balloon is directed from the balloon toward the floor attachment. Its components are

$$\mathbf{T}_{\text{bal}} = -T_{\text{bal}} \sin \theta_{\text{bal}} \hat{\mathbf{x}} - T_{\text{bal}} \cos \theta_{\text{bal}} \hat{\mathbf{y}}.$$

Equilibrium requires

$$\mathbf{T}_{\text{bal}} + \mathbf{F}_{\text{free}} = \mathbf{0},$$

so that

$$T_{\text{bal}} \sin \theta_{\text{bal}} = (\rho V - m)A,$$

and

$$T_{\text{bal}} \cos \theta_{\text{bal}} = (\rho V - m)g.$$

Dividing these equations gives

$$\tan \theta_{\text{bal}} = \frac{A}{g},$$

hence

$$\theta_{\text{bal}} = \tan^{-1}\left(\frac{A}{g}\right).$$

Squaring and adding gives

$$T_{\text{bal}}^2 = (\rho V - m)^2(A^2 + g^2),$$

and, since $\rho V - m > 0$,

$$T_{\text{bal}} = (\rho V - m)\sqrt{g^2 + A^2}.$$

If

$$B_0 = \rho V g$$

is the ordinary buoyant force in a non-accelerating cabin, then

$$\rho V - m = \frac{B_0 - mg}{g}.$$

The balloon tension can consequently also be written as

$$T_{\text{bal}} = (B_0 - mg)\sqrt{1 + \frac{A^2}{g^2}}.$$

If $T_0 = B_0 - mg$ is its floor-tether tension when $A = 0$, then

$$T_{\text{bal}} = T_0 \sqrt{1 + \frac{A^2}{g^2}}.$$

Thus the total balloon mass m by itself does not determine its tension. The displaced volume V , the air density ρ , the ordinary buoyant force B_0 , or the stationary-cabin tension T_0 must additionally be known.

4. The limiting cases, dimensions, and physical directions provide checks on the results. When $A \rightarrow 0$,

$$\theta_{\text{b}}, \theta_{\text{bal}} \rightarrow 0,$$

and

$$T_{\text{b}} \rightarrow mg, \quad T_{\text{bal}} \rightarrow (\rho V - m)g = \rho V g - mg.$$

These are precisely the ordinary vertical-equilibrium results.

When $A \gg g$,

$$\tan \theta \simeq \frac{A}{g} \gg 1, \quad \theta \rightarrow \frac{\pi}{2},$$

so both strings become nearly horizontal. The bob lies backward because the effective gravity points backward, while the balloon lies forward because its net buoyancy points opposite to the effective gravity.

The dimensions are

$$\left[m\sqrt{g^2 + A^2}\right] = [\text{mass}][\text{acceleration}] = [\text{force}],$$

and similarly

$$\left[(\rho V - m)\sqrt{g^2 + A^2}\right] = [\text{force}].$$

Thus both tensions have the required SI unit of newton.

If $\rho V = m$, the pressure force exactly balances the balloon's weight and pseudo-force:

$$\mathbf{F}_{\text{free}} = \mathbf{0}, \quad T_{\text{bal}} = 0.$$

The tether is then slack and its equilibrium angle is not determined. If $\rho V < m$, the free resultant is downward and backward. A floor tether, which can only pull the balloon toward the floor, cannot balance that resultant with a taut string in the stated configuration.

CONCLUSION:

For the bob suspended from the roof,

$$T_b = m\sqrt{g^2 + A^2} \text{ N}, \quad \theta_b = \tan^{-1}\left(\frac{A}{g}\right).$$

The angle is dimensionless and may be expressed in radians or degrees. The bob and its string slope backward, opposite to the direction of aircraft acceleration.

For a balloon of total mass m and displaced volume V , in cabin air of density ρ , a taut floor-tether equilibrium requires

$$\rho V > m.$$

Under this condition,

$$T_{\text{bal}} = (\rho V - m)\sqrt{g^2 + A^2} \text{ N}, \quad \theta_{\text{bal}} = \tan^{-1}\left(\frac{A}{g}\right).$$

Equivalently,

$$T_{\text{bal}} = (B_0 - mg)\sqrt{1 + \frac{A^2}{g^2}} \text{ N}, \quad B_0 = \rho V g.$$

The balloon and its string slope forward, in the direction of aircraft acceleration. Thus the bob and the buoyant balloon make equal angular magnitudes with their respective vertical reference lines, but they lean in opposite horizontal directions. The balloon tension is not determined by the stated mass m alone; its displaced-air mass ρV , ordinary buoyant force B_0 , or stationary-cabin tether tension must also be specified.



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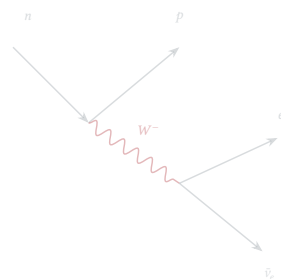
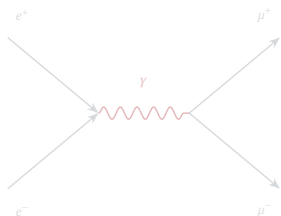
$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$E_n = -\frac{13.6}{n^2} \text{ eV}$$

PREVIOUS YEAR QUESTIONS



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SUBJECT

Waves and Optics

10 SOLVED QUESTIONS • CSE + IFoS

$$n \rightarrow p + e^- + \bar{\nu}_e$$

$$E^2 = p^2 c^2 + m^2 c^4$$

$$\partial_\mu j^\mu = 0$$

SUBJECT-WISE • TOPIC-WISE • SOURCE-TAGGED

For Civil Services and Indian Forest Service Examinations

Sample Preview extbar 10 Solved Questions

Question Index

No.	Question	Marks	Year	Exam	Page
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2	$m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = 0$; A harmonic oscillator is represented by the equation $m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = 0$; where $m = 0.25 \text{ kg}$, $\gamma = 0.07 \text{ kg s}^{-1}$ and $k = 85 \text{ Nm}^{-1}$. Determine (i) the period of oscillation, and (ii) the number of oscillations in which its amplitude will become half of its original value.	15	2025	CSE	4
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4	A particle of mass m is suspended from a spring which has mass m and force constant k . Show that the oscillation frequency is given by $\omega = \frac{\sqrt{3}}{2} \omega_0$, where ω_0 is frequency of oscillation when spring is considered massless.	15	2025	IFoS	7
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CSE • 2026 • 10 MARKS

Q1. A particle executes simple harmonic motion of amplitude A and angular frequency ω . A damping force proportional to velocity acts on the particle. Derive the expression for the displacement of the particle as a function of time and discuss the effect of damping on amplitude and frequency. Also, calculate the time at which the amplitude reduces to half its initial value, if the damping coefficient is b and mass is m .

INTRODUCTION:

Let $x(t)$ denote the displacement from equilibrium, with the positive x -direction chosen arbitrarily. The undamped angular frequency is ω , so the restoring force constant is

$$k = m\omega^2.$$

The velocity-dependent damping force is assumed to be linear and opposite to the instantaneous velocity:

$$F_d = -b\dot{x}, \quad b > 0.$$

Thus the motion is governed by Newton's second law applied in an inertial frame. Define the damping rate

$$\gamma = \frac{b}{2m}.$$

The initial amplitude A will mean the value of the exponentially decaying amplitude envelope at $t = 0$. The phase of the oscillator at that instant is denoted by ϕ .

SOLUTION:

1. The restoring and damping forces give

$$m\ddot{x} = -m\omega^2 x - b\dot{x},$$

or

$$\ddot{x} + \frac{b}{m}\dot{x} + \omega^2 x = 0.$$

Using $\gamma = b/(2m)$, this becomes

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0.$$

To solve it, take

$$x(t) = e^{rt}.$$

Substitution gives the characteristic equation

$$r^2 + 2\gamma r + \omega^2 = 0,$$

whose roots are

$$r = -\gamma \pm \sqrt{\gamma^2 - \omega^2}.$$

For the usual oscillatory, or underdamped, case

$$\gamma < \omega \quad \Longleftrightarrow \quad b < 2m\omega,$$

define the damped angular frequency

$$\omega_d = \sqrt{\omega^2 - \gamma^2} = \sqrt{\omega^2 - \frac{b^2}{4m^2}}.$$

The roots are then

$$r = -\gamma \pm i\omega_d.$$

Hence

$$x(t) = e^{-\gamma t} (C \cos \omega_d t + D \sin \omega_d t).$$

Writing $C = A \cos \phi$ and $D = -A \sin \phi$, the displacement is

$$x(t) = Ae^{-bt/(2m)} \cos(\omega_d t + \phi)$$

$$\omega_d = \sqrt{\omega^2 - \frac{b^2}{4m^2}}.$$

If the initial conditions are specified as $x(0) = A$ and $\dot{x}(0) = 0$, direct determination of the constants instead gives

$$x(t) = Ae^{-bt/(2m)} \left[\cos(\omega_d t) + \frac{b}{2m\omega_d} \sin(\omega_d t) \right].$$

This is equivalent to the amplitude-phase form with an appropriate constant phase and amplitude parameter.

In the zero-damping limit $b \rightarrow 0$,

$$\gamma \rightarrow 0, \quad \omega_d \rightarrow \omega,$$

and the solution reduces to ordinary simple harmonic motion, as required.

2. The time-dependent amplitude envelope is

$$A(t) = Ae^{-\gamma t} = Ae^{-bt/(2m)}.$$

Therefore damping causes the amplitude to decrease exponentially rather than remain constant. Since the mechanical energy of a harmonic oscillator is proportional to the square of its amplitude,

$$E(t) \propto A^2(t),$$

its slowly varying envelope decays as

$$E(t) = E(0)e^{-bt/m}.$$

This agrees with the fact that

$$\frac{dE}{dt} = -b\dot{x}^2 \leq 0,$$

so the damping force continuously removes mechanical energy.

Damping also lowers the oscillation frequency from ω to

$$\omega_d = \sqrt{\omega^2 - \frac{b^2}{4m^2}} < \omega.$$

For weak damping, $b \ll 2m\omega$, the binomial expansion gives

$$\omega_d = \omega \sqrt{1 - \frac{b^2}{4m^2\omega^2}} \approx \omega \left(1 - \frac{b^2}{8m^2\omega^2}\right).$$

Thus the reduction in frequency is second order in b for weak damping, whereas the amplitude-decay rate is first order in b .

At critical damping,

$$b = 2m\omega,$$

the two characteristic roots coincide and

$$x(t) = (C + Dt)e^{-\omega t}.$$

The particle returns to equilibrium without oscillating. For

$$b > 2m\omega,$$

the motion is overdamped:

$$x(t) = Ce^{(-\gamma + \sqrt{\gamma^2 - \omega^2})t} + De^{(-\gamma - \sqrt{\gamma^2 - \omega^2})t},$$

and again no oscillation frequency exists.

3. Let $t_{1/2}$ be the time at which the amplitude envelope becomes $A/2$. Then

$$Ae^{-bt_{1/2}/(2m)} = \frac{A}{2}.$$

Cancelling A and taking natural logarithms,

$$-\frac{bt_{1/2}}{2m} = \ln\left(\frac{1}{2}\right) = -\ln 2.$$

Therefore

$$t_{1/2} = \frac{2m}{b} \ln 2.$$

Since b/m has dimensions of inverse time, $2m/b$ has dimensions of time, as required.

CONCLUSION:

For underdamped motion, $b < 2m\omega$, the displacement is

$$x(t) = Ae^{-bt/(2m)} \cos \left[\sqrt{\omega^2 - \frac{b^2}{4m^2}} t + \phi \right].$$

The amplitude decreases exponentially according to

$$A(t) = Ae^{-bt/(2m)},$$

while the angular frequency is reduced from the undamped value ω to

$$\omega_d = \sqrt{\omega^2 - \frac{b^2}{4m^2}}.$$

At $b = 2m\omega$ the system is critically damped, and for $b > 2m\omega$ it is overdamped; in both cases the motion is non-oscillatory. The time required for the amplitude envelope to fall to one-half of its initial value is

$$t_{1/2} = \frac{2m \ln 2}{b},$$

measured in seconds when m is in kilograms and b is in kg s^{-1} .

CSE • 2025 • 15 MARKS

Q2. $m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = 0$; A harmonic oscillator is represented by the equation $m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = 0$; where $m = 0.25 \text{ kg}$, $\gamma = 0.07 \text{ kg s}^{-1}$ and $k = 85 \text{ Nm}^{-1}$. Determine (i) the period of oscillation, and (ii) the number of oscillations in which its amplitude will become half of its original value.

INTRODUCTION:

The equation describes a one-dimensional oscillator subject to a linear restoring force and viscous damping. The displacement x is measured from the equilibrium position along a fixed positive axis. The restoring force is $-kx$, while the damping force is

$$F_d = -\gamma \dot{x},$$

so it always acts opposite to the instantaneous velocity.

The motion is assumed to remain in the linear regime, with constant m , k , and γ , and with no external driving force. Define

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad \beta = \frac{\gamma}{2m},$$

where ω_0 is the undamped natural angular frequency and β is the amplitude-decay constant. For underdamped motion, $\beta < \omega_0$, and the damped angular frequency is

$$\omega_d = \sqrt{\omega_0^2 - \beta^2}.$$

The period is $T = 2\pi/\omega_d$. The number of oscillations required for the exponentially decaying amplitude to

become one-half is obtained from the amplitude envelope.

SOLUTION:

Dividing the equation of motion by m ,

$$\ddot{x} + \frac{\gamma}{m}\dot{x} + \frac{k}{m}x = 0.$$

Using

$$2\beta = \frac{\gamma}{m}, \quad \omega_0^2 = \frac{k}{m},$$

this becomes

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2x = 0.$$

For a trial solution $x = e^{rt}$, the characteristic equation is

$$r^2 + 2\beta r + \omega_0^2 = 0.$$

Hence

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}.$$

If $\beta < \omega_0$, the roots may be written as

$$r = -\beta \pm i\omega_d, \quad \omega_d = \sqrt{\omega_0^2 - \beta^2}.$$

The displacement is therefore

$$x(t) = A_0 e^{-\beta t} \cos(\omega_d t + \phi),$$

where A_0 and ϕ are fixed by the initial conditions. Thus the amplitude envelope is

$$A(t) = A_0 e^{-\beta t}.$$

The units are consistent:

$$\left[\frac{k}{m} \right] = \frac{\text{N m}^{-1}}{\text{kg}} = \frac{\text{kg s}^{-2}}{\text{kg}} = \text{s}^{-2},$$

and

$$\left[\frac{\gamma}{m} \right] = \frac{\text{kg s}^{-1}}{\text{kg}} = \text{s}^{-1}.$$

(i) The decay constant is

$$\beta = \frac{\gamma}{2m} = \frac{0.07 \text{ kg s}^{-1}}{2(0.25 \text{ kg})} = \frac{0.07}{0.50} \text{ s}^{-1} = 0.14 \text{ s}^{-1}.$$

The square of the undamped natural angular frequency is

$$\omega_0^2 = \frac{k}{m} = \frac{85 \text{ N m}^{-1}}{0.25 \text{ kg}} = 340 \text{ s}^{-2},$$

so that

$$\omega_0 = \sqrt{340} \text{ rad s}^{-1} = 18.439 \text{ rad s}^{-1}.$$

Since

$$\beta^2 = (0.14)^2 = 0.0196 \text{ s}^{-2} < 340 \text{ s}^{-2} = \omega_0^2,$$

the oscillator is strongly underdamped and executes oscillatory motion.

Its damped angular frequency is

$$\begin{aligned} \omega_d &= \sqrt{\omega_0^2 - \beta^2} \\ &= \sqrt{340 - 0.0196} \text{ rad s}^{-1} \\ &= \sqrt{339.9804} \text{ rad s}^{-1} \\ &= 18.4386 \text{ rad s}^{-1}. \end{aligned}$$

Therefore, the period of the damped oscillation is

$$\begin{aligned} T &= \frac{2\pi}{\omega_d} \\ &= \frac{2\pi}{18.4386 \text{ rad s}^{-1}} \\ &= 0.34076 \text{ s} \\ &\approx 0.341 \text{ s}. \end{aligned}$$

As a limiting check, when $\gamma \rightarrow 0$, one has $\beta \rightarrow 0$ and

$$T \rightarrow T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{18.439} \text{ s} \approx 0.34075 \text{ s}.$$

The computed damped period is very slightly greater than T_0 , as expected because damping reduces the angular frequency.

(ii) Let $t_{1/2}$ be the time at which the amplitude becomes one-half of its initial value. From

$$A(t) = A_0 e^{-\beta t},$$

the condition $A(t_{1/2}) = A_0/2$ gives

$$A_0 e^{-\beta t_{1/2}} = \frac{A_0}{2}.$$

After cancelling A_0 ,

$$e^{-\beta t_{1/2}} = \frac{1}{2}.$$

Taking natural logarithms,

$$-\beta t_{1/2} = -\ln 2,$$

and hence

$$\begin{aligned} t_{1/2} &= \frac{\ln 2}{\beta} \\ &= \frac{0.693147}{0.14 \text{ s}^{-1}} \\ &= 4.951 \text{ s}. \end{aligned}$$

If N denotes the number of oscillation periods completed in this time, then

$$N = \frac{t_{1/2}}{T}.$$

Therefore,

$$N = \frac{4.951 \text{ s}}{0.34076 \text{ s}} = 14.53.$$

Equivalently,

$$N = \frac{\omega_d \ln 2}{2\pi\beta} = \frac{(18.4386)(0.693147)}{2\pi(0.14)} = 14.53.$$

For a discrete-cycle plausibility check, after 14 complete oscillations,

$$\frac{A_{14}}{A_0} = e^{-\beta(14T)} = e^{-(0.14)(14)(0.34076)} \approx 0.513,$$

whereas after 15 complete oscillations,

$$\frac{A_{15}}{A_0} = e^{-\beta(15T)} = e^{-(0.14)(15)(0.34076)} \approx 0.489.$$

Thus the amplitude crosses $A_0/2$ between the fourteenth and fifteenth completed oscillations, consistently with $N = 14.53$.

CONCLUSION:

The oscillator is underdamped because

$$\beta = 0.14 \text{ s}^{-1} < \omega_0 = 18.439 \text{ rad s}^{-1}.$$

Its damped angular frequency and period are

$$\boxed{\omega_d = 18.4386 \text{ rad s}^{-1}}, \quad \boxed{T \approx 0.341 \text{ s}}.$$

The amplitude decays as

$$A(t) = A_0 e^{-0.14t},$$

because the damping force acts opposite to the velocity. It becomes half of its initial value after

$$\boxed{t_{1/2} \approx 4.95 \text{ s}},$$

corresponding to

$$\boxed{N \approx 14.53 \text{ oscillations}}.$$

Hence the amplitude becomes one-half between the fourteenth and fifteenth completed oscillations; if an integral number of completed oscillations is required, it is below one-half after approximately

$$\boxed{15 \text{ complete oscillations}}.$$

is 250 m/s, and the speed of sound in air is 350 m/s. What is the frequency and wavelength of the emitted sound wave?

INTRODUCTION:

The required quantities are the frequency f_{sound} and wavelength λ_{sound} of the sound emitted into air.

Let

$$L = 15.0 \text{ cm} = 0.150 \text{ m}$$

$$v_s = 250 \text{ m s}^{-1}$$

$$v_a = 350 \text{ m s}^{-1}$$

where v_s is the transverse-wave speed on the string and v_a is the speed of sound in air.

Choose the x -axis along the string, with $x = 0$ at the left fixed end and $x = L$ at the right fixed end. The transverse displacement is denoted by $y(x, t)$, with positive y upward. Since both ends are fixed,

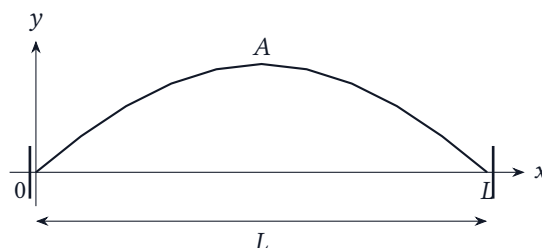
$$y(0, t) = y(L, t) = 0.$$

Thus, both ends must be nodes of every allowed standing-wave mode. The largest allowed wavelength is therefore the wavelength of the fundamental mode, for which the string contains one half-wavelength.

It is assumed that the string is uniform, the oscillations are small, the air is stationary and approximately nondispersive, and losses and end corrections are negligible. The string forces the adjacent air to oscillate at the same frequency; hence the sound frequency equals the vibration frequency of the string, although its wavelength changes because the wave speed changes on passing from the string to air.

SOLUTION:

For the fundamental standing-wave mode, the fixed ends are consecutive nodes and there is one antinode at $x = L/2$:



In general, if n half-wavelengths fit into a string of length L ,

$$L = n \frac{\lambda_n}{2},$$

so that

$$\lambda_n = \frac{2L}{n}, \quad n = 1, 2, 3, \dots$$

Q3. A 15.0 cm violin string, fixed at both the ends, is vibrating with the largest wavelength. The speed of the wave in this string

The wavelength is largest for the smallest permitted positive integer, $n = 1$. Hence the fundamental wavelength on the string is

$$\lambda_s = 2L.$$

Substituting the string length,

$$\lambda_s = 2(0.150 \text{ m}) = 0.300 \text{ m}.$$

1. The frequency of vibration of the string is obtained from $v = f\lambda$:

$$f_s = \frac{v_s}{\lambda_s} = \frac{250 \text{ m s}^{-1}}{0.300 \text{ m}} = 8.33 \times 10^2 \text{ s}^{-1}.$$

Therefore,

$$f_s = 8.33 \times 10^2 \text{ Hz}.$$

The air is driven periodically by the string. Since the number of oscillations produced per second cannot change at the string-air coupling,

$$f_{\text{sound}} = f_s = 8.33 \times 10^2 \text{ Hz}.$$

2. The wavelength of the sound in air is

$$\lambda_{\text{sound}} = \frac{v_a}{f_{\text{sound}}}.$$

On substitution,

$$\lambda_{\text{sound}} = \frac{350 \text{ m s}^{-1}}{8.33 \times 10^2 \text{ s}^{-1}} = 4.20 \times 10^{-1} \text{ m}.$$

Thus,

$$\lambda_{\text{sound}} = 0.420 \text{ m} = 42.0 \text{ cm}.$$

The dimensional checks are

$$[f] = \frac{[v]}{[\lambda]} = \frac{\text{m s}^{-1}}{\text{m}} = \text{s}^{-1} = \text{Hz}$$

$$[\lambda_{\text{sound}}] = \frac{[v_a]}{[f]} = \frac{\text{m s}^{-1}}{\text{s}^{-1}} = \text{m}.$$

The result is also physically plausible: the frequency remains unchanged, while

$$\frac{\lambda_{\text{sound}}}{\lambda_s} = \frac{v_a}{v_s} = \frac{350}{250} = 1.40.$$

Therefore, because sound travels 1.40 times faster in air than the transverse wave travels on the string, its wavelength in air is 1.40 times the standing-wave wavelength on the string:

$$1.40(0.300 \text{ m}) = 0.420 \text{ m}.$$

CONCLUSION:

For the largest-wavelength, fixed-end mode, the string vibrates in its fundamental mode with

$$\lambda_s = 2L = 0.300 \text{ m}.$$

The emitted sound has frequency

$$f_{\text{sound}} = 8.33 \times 10^2 \text{ Hz} \approx 833 \text{ Hz}$$

and wavelength in air

$$\lambda_{\text{sound}} = 0.420 \text{ m} = 42.0 \text{ cm}.$$

The sound propagates outward through the air from the vibrating string; no unique propagation direction is specified by the question. These results apply for a uniform string vibrating in the fundamental mode and for stationary, approximately nondispersive air with negligible end corrections and losses. The physical inference is that coupling to air preserves the frequency, while the higher wave speed in air produces a correspondingly larger wavelength.

IFoS • 2025 • 15 MARKS

Q4. A particle of mass m is suspended from a spring which has mass m and force constant k . Show that the oscillation frequency is given by $\omega = \frac{\sqrt{3}}{2}\omega_0$, where ω_0 is frequency of oscillation when spring is considered massless.

INTRODUCTION:

The required quantity is the angular frequency of small vertical oscillations when the inertia of the spring is included. Let the suspended particle have mass M , let the uniform spring have mass m_s , natural or equilibrium-coordinate length L , and force constant k . At the end we shall use the data

$$M = m_s = m.$$

Let y , measured vertically downward from the fixed end, label a point of the spring, so that $0 \leq y \leq L$. Let $x(t)$ be the small downward displacement of the lower end, and hence of the suspended particle, from the static equilibrium position. Measuring displacement from static equilibrium removes all terms linear in x arising from gravity; gravity changes the equilibrium extension but not the small-oscillation frequency.

For the standard effective-mass treatment of a uniform spring, the incremental longitudinal extension

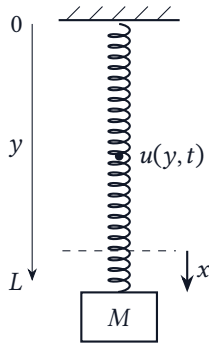
is assumed to be distributed uniformly. Since the upper end is fixed and the lower end moves through $x(t)$, the displacement field is

$$u(y, t) = \frac{y}{L}x(t), \quad u(0, t) = 0, \quad u(L, t) = x(t).$$

Thus different parts of the massive spring have different velocities and the entire spring mass cannot be added directly to M . The governing principle is conservation of mechanical energy, or equivalently Lagrange's equation, with both the particle's kinetic energy and the distributed kinetic energy of the spring included.

SOLUTION:

The geometry and sign convention are shown below. The coordinate y and the displacement x are both positive vertically downward.



Since the spring is uniform, its linear mass density is

$$\lambda = \frac{m_s}{L}.$$

An element of length dy therefore has mass

$$dm = \lambda dy = \frac{m_s}{L} dy.$$

From the assumed displacement field,

$$u(y, t) = \frac{y}{L}x(t),$$

the velocity of this element is

$$v(y, t) = \frac{\partial u}{\partial t} = \frac{y}{L}\dot{x}(t).$$

The element at the fixed end has zero velocity, while the element at the lower end has velocity $\dot{x}$, as required by the boundary conditions.

The kinetic energy of the spring is consequently

$$T_s = \frac{1}{2} \int_0^L v^2 dm.$$

Substituting $v = (y/L)\dot{x}$ and $dm = (m_s/L)dy$,

$$\begin{aligned} T_s &= \frac{1}{2} \int_0^L \left(\frac{y}{L} \dot{x} \right)^2 \frac{m_s}{L} dy \\ &= \frac{m_s \dot{x}^2}{2L^3} \int_0^L y^2 dy \\ &= \frac{m_s \dot{x}^2}{2L^3} \left[\frac{y^3}{3} \right]_0^L \\ &= \frac{m_s \dot{x}^2}{2L^3} \frac{L^3}{3} = \frac{1}{2} \left(\frac{m_s}{3} \right) \dot{x}^2. \end{aligned}$$

Thus, for this mode of motion, the spring contributes the effective inertial mass

$$m_{\text{eff, spring}} = \frac{m_s}{3}.$$

The kinetic energy of the suspended particle is

$$T_M = \frac{1}{2} M \dot{x}^2.$$

Hence the total kinetic energy is

$$T = T_M + T_s = \frac{1}{2} \left(M + \frac{m_s}{3} \right) \dot{x}^2.$$

The elastic potential energy can also be obtained directly from the distributed deformation. A spring segment of equilibrium length dy has force constant

$$k_y = k \frac{L}{dy},$$

because the force constant of a uniform spring is inversely proportional to its length. Its incremental extension is

$$du = \frac{\partial u}{\partial y} dy = \frac{x}{L} dy.$$

Therefore its elastic energy is

$$dV = \frac{1}{2} k_y (du)^2 = \frac{1}{2} \left(k \frac{L}{dy} \right) \left(\frac{x}{L} dy \right)^2 = \frac{1}{2} \frac{kx^2}{L} dy.$$

Integrating over the spring gives

$$V = \int_0^L dV = \frac{1}{2} \frac{kx^2}{L} \int_0^L dy = \frac{1}{2} kx^2.$$

The Lagrangian is therefore

$$\mathcal{L} = \frac{1}{2} \left(M + \frac{m_s}{3} \right) \dot{x}^2 - \frac{1}{2} kx^2.$$

Lagrange's equation,

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{\partial \mathcal{L}}{\partial x} = 0,$$

gives

$$\left(M + \frac{m_s}{3} \right) \ddot{x} + kx = 0.$$

Comparing this with the simple-harmonic equation

$$\ddot{x} + \omega^2 x = 0,$$

we obtain the general result

$$\omega^2 = \frac{k}{M + m_s/3}, \quad \omega = \sqrt{\frac{k}{M + m_s/3}}.$$

In the present problem,

$$M = m_s = m,$$

and hence

$$\begin{aligned} \omega &= \sqrt{\frac{k}{m + m/3}} \\ &= \sqrt{\frac{k}{4m/3}} \\ &= \sqrt{\frac{3k}{4m}} \\ &= \frac{\sqrt{3}}{2} \sqrt{\frac{k}{m}}. \end{aligned}$$

When the spring is considered massless, $m_s = 0$, so the angular frequency is

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

Consequently,

$$\omega = \frac{\sqrt{3}}{2} \omega_0.$$

The dimensional check is

$$\left[\frac{k}{m} \right] = \frac{\text{N m}^{-1}}{\text{kg}} = \frac{\text{kg s}^{-2}}{\text{kg}} = \text{s}^{-2},$$

so ω has dimension s^{-1} , conventionally expressed in rad s^{-1} .

As a limiting check, the general expression satisfies

$$\lim_{m_s \rightarrow 0} \sqrt{\frac{k}{M + m_s/3}} = \sqrt{\frac{k}{M}},$$

which is exactly the massless-spring result. Also,

$$\frac{\omega}{\omega_0} = \frac{\sqrt{3}}{2} \approx 0.866 < 1,$$

which is physically plausible: the moving spring adds inertia without increasing the force constant, and therefore lowers the frequency. During the oscillation the energy

$$E = \frac{1}{2} \left(M + \frac{m_s}{3} \right) \dot{x}^2 + \frac{1}{2} kx^2$$

is conserved.

CONCLUSION:

For small vertical oscillations of a uniform linear spring, with the upper end fixed and the incremental displacement varying as $u(y, t) = yx(t)/L$, the spring's distributed mass m_s contributes an effective oscillating mass $m_s/3$. Therefore

$$\omega = \sqrt{\frac{k}{M + m_s/3}}.$$

For a suspended particle and spring each of mass m ,

$$\omega = \sqrt{\frac{3k}{4m}} = \frac{\sqrt{3}}{2} \omega_0, \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

The motion is simple harmonic in the vertical direction about the static equilibrium position, and ω is measured in rad s^{-1} . The result is valid for small longitudinal oscillations of a uniform Hookean spring under the linear displacement-profile approximation. It shows that the inertia of the spring reduces the angular frequency to approximately 86.6% of its massless-spring value.

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Q5. A weakly damped harmonic oscillator consisting of spring-mass system has the following parameters : Mass $m = 0.25 \text{ kg}$, Spring constant $k = 100 \text{ N m}^{-1}$, Damping coefficient $\gamma = 1 \text{ N s m}^{-1}$. A periodic force $F = 5 \cos \omega t$ (newton) is applied to the system. Determine (i) the amplitude of the oscillator at resonance and (ii) the Q -value of the oscillator.

INTRODUCTION:

The required quantities are the maximum steady-state displacement amplitude under harmonic forcing and the quality factor of the oscillator. Let $x(t)$ denote displacement along the spring axis, with positive x chosen in the direction of the applied force at $t = 0$. The equation of motion is

$$m\ddot{x} + \gamma\dot{x} + kx = F_0 \cos \omega t, \quad F_0 = 5 \text{ N}.$$

Here the damping force is $-\gamma\dot{x}$, so that it always opposes the velocity.

Define the undamped natural angular frequency and the damping parameter by

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad \beta = \frac{\gamma}{2m}.$$

The weak-damping assumption requires

$$\beta \ll \omega_0 \iff \gamma \ll 2\sqrt{mk}.$$

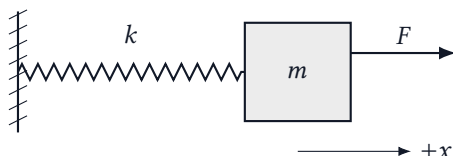
The steady-state displacement is written as

$$x(t) = A(\omega) \cos(\omega t - \delta),$$

where $A(\omega)$ is the non-negative amplitude and δ is the displacement lag relative to the applied force. Transient motion is omitted because it decays as $e^{-\beta t}$.

SOLUTION:

The physical arrangement and the adopted positive direction are represented schematically below.



Substitution of

$$x(t) = A \cos(\omega t - \delta)$$

into the equation of motion, or equivalently use of complex amplitudes, gives

$$(k - m\omega^2 + i\gamma\omega) \tilde{x} = F_0.$$

Therefore,

$$A(\omega) = |\tilde{x}| = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \gamma^2\omega^2}},$$

and the phase lag satisfies

$$\tan \delta = \frac{\gamma\omega}{k - m\omega^2}.$$

1. For displacement resonance, $A(\omega)$ is maximized. Since F_0 is fixed, it is sufficient to minimize

$$D(\omega) = (k - m\omega^2)^2 + \gamma^2\omega^2.$$

Differentiating,

$$\frac{dD}{d\omega} = 2(k - m\omega^2)(-2m\omega) + 2\gamma^2\omega.$$

For the non-zero resonant frequency,

$$-4m(k - m\omega_r^2) + 2\gamma^2 = 0,$$

so that

$$k - m\omega_r^2 = \frac{\gamma^2}{2m}.$$

Hence

$$\omega_r^2 = \frac{k}{m} - \frac{\gamma^2}{2m^2} = \omega_0^2 - 2\beta^2.$$

For the given data,

$$\omega_0 = \sqrt{\frac{100 \text{ N m}^{-1}}{0.25 \text{ kg}}} = \sqrt{400 \text{ s}^{-2}} = 20.0 \text{ rad s}^{-1},$$

and

$$\beta = \frac{1 \text{ N s m}^{-1}}{2(0.25 \text{ kg})} = 2.00 \text{ s}^{-1}.$$

Thus

$$\omega_r = \sqrt{(20.0)^2 - 2(2.00)^2} \text{ rad s}^{-1} = \sqrt{392} \text{ rad s}^{-1} = 19.8 \text{ rad s}^{-1}.$$

At this frequency,

$$k - m\omega_r^2 = \frac{\gamma^2}{2m} = \frac{(1 \text{ N s m}^{-1})^2}{2(0.25 \text{ kg})} = 2.00 \text{ N m}^{-1}.$$

Therefore,

$$\begin{aligned} A_r &= \frac{F_0}{\sqrt{(k - m\omega_r^2)^2 + \gamma^2\omega_r^2}} \\ &= \frac{5 \text{ N}}{\sqrt{(2.00 \text{ N m}^{-1})^2 + (1 \text{ N s m}^{-1})^2(19.8 \text{ s}^{-1})^2}} \\ &= \frac{5 \text{ N}}{\sqrt{4.00 + 392 \text{ N m}^{-1}}} \\ &= \frac{5}{19.9} \text{ m} \\ &= 0.251 \text{ m}. \end{aligned}$$

Equivalently, using the resonance condition,

$$A_r = \frac{F_0}{\gamma\sqrt{\omega_0^2 - \beta^2}} = \frac{F_0}{\gamma\sqrt{\omega_0^2 - \gamma^2/(4m^2)}}.$$

Under the weak-damping approximation $\beta^2 \ll \omega_0^2$,

$$A_r \simeq \frac{F_0}{\gamma\omega_0} = \frac{5 \text{ N}}{(1 \text{ N s m}^{-1})(20.0 \text{ s}^{-1})} = 0.250 \text{ m}.$$

Thus the exact displacement-resonance result and the usual weak-damping result agree to within about 0.5%.

The dimensional check is

$$[\gamma\omega_0] = (\text{N s m}^{-1})(\text{s}^{-1}) = \text{N m}^{-1},$$

and hence

$$\left[\frac{F_0}{\gamma\omega_0} \right] = \frac{\text{N}}{\text{N m}^{-1}} = \text{m},$$

as required for a displacement amplitude.

A useful physical check is the limit $\gamma \rightarrow 0$: the resonant amplitude varies as $1/\gamma$ and therefore becomes unbounded in the ideal undamped steady-state model. This is the expected resonance behaviour. At the actual displacement-resonance frequency,

$$\tan \delta_r = \frac{\gamma\omega_r}{\gamma^2/(2m)} = \frac{2m\omega_r}{\gamma} \simeq 9.90,$$

so

$$\delta_r \simeq 84.2^\circ.$$

The displacement therefore lags the force by nearly 90° , as expected for weak damping near resonance.

2. The quality factor of a weakly damped oscillator is

$$Q = 2\pi \frac{\text{energy stored}}{\text{energy lost per cycle}} = \frac{\omega_0}{2\beta} = \frac{m\omega_0}{\gamma} = \frac{\sqrt{mk}}{\gamma}.$$

Substitution gives

$$\begin{aligned} Q &= \frac{(0.25 \text{ kg})(20.0 \text{ s}^{-1})}{1 \text{ N s m}^{-1}} \\ &= \frac{5.00 \text{ kg s}^{-1}}{1 \text{ kg s}^{-1}} \\ &= 5.00. \end{aligned}$$

Thus Q is dimensionless.

The same result follows from the fractional energy loss. For weak damping, the mechanical energy decays as

$$E(t) \propto e^{-2\beta t}.$$

During one approximately natural period $T_0 = 2\pi/\omega_0$, the small fractional energy loss is

$$\frac{\Delta E}{E} \simeq 2\beta T_0 = \frac{4\pi\beta}{\omega_0}.$$

Consequently,

$$Q = 2\pi \frac{E}{\Delta E} \simeq \frac{\omega_0}{2\beta} = 5.00,$$

which confirms the result.

The weak-damping condition is also numerically consistent:

$$\begin{aligned} \frac{\beta}{\omega_0} &= \frac{2.00}{20.0} = 0.100 \\ \frac{\gamma}{2\sqrt{mk}} &= \frac{1}{2\sqrt{(0.25)(100)}} = 0.100. \end{aligned}$$

Moreover, $\omega_r^2 = \omega_0^2 - 2\beta^2 > 0$, so a non-zero displacement-resonance peak exists.

CONCLUSION:

For the convention

$$m\ddot{x} + \gamma\dot{x} + kx = 5 \cos \omega t,$$

the displacement-resonance frequency is

$$\omega_r = 19.8 \text{ rad s}^{-1}.$$

The corresponding steady-state amplitude along the spring axis is

$$A_r = 0.251 \text{ m},$$

or, within the standard weak-damping approximation,

$$A_r \simeq \frac{F_0}{\gamma\omega_0} = 0.250 \text{ m}.$$

The displacement lags the driving force by approximately

$$\delta_r \simeq 84.2^\circ,$$

which is close to the expected 90° lag at weakly damped resonance.

The oscillator quality factor is

$$Q = \frac{m\omega_0}{\gamma} = \frac{\sqrt{mk}}{\gamma} = 5.00,$$

which is dimensionless. These results apply to the linear steady-state motion with viscous damping, after the transient has decayed. The value $Q = 5$ indicates a moderately sharp resonance, while the conditions $\beta/\omega_0 = 0.10$ and $\omega_r^2 > 0$ confirm that the assumed weakly damped resonance is physically admissible.

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Q6. Consider a damped harmonic oscillator given by the equation $\ddot{x} + 2\mu\dot{x} + \omega_0^2 x = 0$. Obtain the displacements for (i) overdamped, (ii) underdamped and, (iii) critically damped motion.

INTRODUCTION:

The required displacement is obtained by solving the homogeneous, linear equation of motion

$$\ddot{x} + 2\mu\dot{x} + \omega_0^2 x = 0.$$

Here $x(t)$ is the signed displacement from the stable equilibrium position, measured along a fixed x -axis; $x > 0$ and $x < 0$ denote the two opposite directions along this axis. The motion is described in an inertial frame. The parameter $\omega_0 > 0$ is the natural angular frequency of the corresponding undamped oscillator, while $\mu \geq 0$ is the damping constant per unit mass. Thus

$$[\mu] = [\omega_0] = T^{-1}.$$

The equation assumes linear restoring force, viscous damping proportional to velocity, constant coefficients, and the absence of an external driving force.

For a mechanical oscillator of mass m , spring constant k , and viscous coefficient b ,

$$\omega_0^2 = \frac{k}{m}, \quad 2\mu = \frac{b}{m},$$

so the three regimes are distinguished by comparing the damping rate μ with the natural frequency ω_0 . For completeness, let

$$x(0) = x_0, \quad \dot{x}(0) = v_0,$$

where x_0 and v_0 are arbitrary initial displacement and velocity. The general solutions and their corresponding initial-value forms will both be obtained.

SOLUTION:

Since the coefficients are constant, take the trial displacement

$$x(t) = e^{\lambda t}.$$

Then

$$\dot{x} = \lambda e^{\lambda t}, \quad \ddot{x} = \lambda^2 e^{\lambda t}.$$

Substitution into the equation of motion gives

$$(\lambda^2 + 2\mu\lambda + \omega_0^2) e^{\lambda t} = 0.$$

Because $e^{\lambda t} \neq 0$, the characteristic equation is

$$\lambda^2 + 2\mu\lambda + \omega_0^2 = 0,$$

whose roots are

$$\lambda_{1,2} = -\mu \pm \sqrt{\mu^2 - \omega_0^2}.$$

The nature of these roots determines the form of the displacement.

- (i) For overdamped motion, $\mu > \omega_0$.

Define

$$\alpha = \sqrt{\mu^2 - \omega_0^2} > 0.$$

The two characteristic roots are real and distinct:

$$\lambda_1 = -\mu + \alpha, \quad \lambda_2 = -\mu - \alpha.$$

Since $\omega_0 > 0$,

$$\alpha = \sqrt{\mu^2 - \omega_0^2} < \mu,$$

and therefore both roots are negative:

$$\lambda_1 < 0, \quad \lambda_2 < 0.$$

Hence the general displacement is

$$x(t) = C_1 e^{(-\mu+\alpha)t} + C_2 e^{(-\mu-\alpha)t}$$

$$\alpha = \sqrt{\mu^2 - \omega_0^2}.$$

Equivalently,

$$x(t) = e^{-\mu t} (C_1 e^{\alpha t} + C_2 e^{-\alpha t}) = e^{-\mu t} (A \cosh \alpha t + B \sinh \alpha t).$$

To express the constants in terms of x_0 and v_0 , use

$$x(0) = A = x_0.$$

Differentiating,

$$\dot{x} = e^{-\mu t} [-\mu (A \cosh \alpha t + B \sinh \alpha t) + \alpha (A \sinh \alpha t + B \cosh \alpha t)].$$

At $t = 0$,

$$v_0 = -\mu A + \alpha B.$$

Thus

$$B = \frac{v_0 + \mu x_0}{\alpha},$$

and the initial-value form is

$$x(t) = e^{-\mu t} \left[x_0 \cosh(\alpha t) + \frac{v_0 + \mu x_0}{\alpha} \sinh(\alpha t) \right], \quad \alpha = \sqrt{\mu^2 - \omega_0^2}.$$

No sinusoidal factor occurs, so there is no sustained oscillation. Depending on the initial conditions, the oscillator can approach equilibrium from one side or cross the equilibrium at most once before approaching it asymptotically. At sufficiently large t , the more slowly decaying mode dominates:

$$x(t) \sim C_1 e^{-(\mu-\alpha)t}.$$

In the strong-damping limit $\mu \gg \omega_0$,

$$\alpha = \mu \sqrt{1 - \frac{\omega_0^2}{\mu^2}} \approx \mu - \frac{\omega_0^2}{2\mu},$$

so

$$\mu - \alpha \approx \frac{\omega_0^2}{2\mu}.$$

Consequently, the slow part of the displacement behaves as

$$x(t) \sim C_1 \exp\left(-\frac{\omega_0^2}{2\mu} t\right),$$

showing that very large damping can make the final return to equilibrium slow.

- (ii) For underdamped motion, $0 \leq \mu < \omega_0$.

Define the damped angular frequency

$$\omega_d = \sqrt{\omega_0^2 - \mu^2} > 0.$$

Then

$$\sqrt{\mu^2 - \omega_0^2} = i\omega_d,$$

and the characteristic roots are the complex conjugate pair

$$\lambda_{1,2} = -\mu \pm i\omega_d.$$

The corresponding real general displacement is

$$x(t) = e^{-\mu t} (C_1 \cos \omega_d t + C_2 \sin \omega_d t), \quad \omega_d = \sqrt{\omega_0^2 - \mu^2}.$$

It may also be written as

$$x(t) = A_0 e^{-\mu t} \cos(\omega_d t - \phi),$$

where A_0 and ϕ are determined by the initial conditions.

Using $x(0) = x_0$ gives

$$C_1 = x_0.$$

Differentiating,

$$\dot{x} = e^{-\mu t} [-\mu(C_1 \cos \omega_d t + C_2 \sin \omega_d t) - \omega_d C_1 \sin \omega_d t + \omega_d C_2 \cos \omega_d t].$$

At $t = 0$,

$$v_0 = -\mu C_1 + \omega_d C_2,$$

and hence

$$C_2 = \frac{v_0 + \mu x_0}{\omega_d}.$$

Therefore,

$$x(t) = e^{-\mu t} \left[x_0 \cos(\omega_d t) + \frac{v_0 + \mu x_0}{\omega_d} \sin(\omega_d t) \right].$$

The motion is oscillatory, but its amplitude is bounded by an exponentially decreasing envelope proportional to

$$e^{-\mu t}.$$

The damped period is

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{\omega_0^2 - \mu^2}}.$$

Since $\omega_d < \omega_0$, damping reduces the oscillation frequency and increases the period:

$$T_d > \frac{2\pi}{\omega_0}.$$

Successive extrema of the same sign, separated by one damped period, have amplitudes in the ratio

$$\frac{|x(t + T_d)|}{|x(t)|} = e^{-\mu T_d},$$

so the logarithmic decrement is

$$\ln \frac{|x(t)|}{|x(t + T_d)|} = \mu T_d = \frac{2\pi\mu}{\omega_d}.$$

The undamped limit supplies an immediate check. When $\mu \rightarrow 0$,

$$\omega_d \rightarrow \omega_0, \quad e^{-\mu t} \rightarrow 1,$$

and the displacement becomes

$$x(t) \rightarrow x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t,$$

which is the standard displacement of an undamped simple harmonic oscillator.

(iii) For critically damped motion, $\mu = \omega_0$.

The characteristic equation becomes

$$\lambda^2 + 2\omega_0 \lambda + \omega_0^2 = 0,$$

or

$$(\lambda + \omega_0)^2 = 0.$$

Thus there is a repeated root

$$\lambda = -\omega_0 = -\mu.$$

For a repeated root of a second-order linear differential equation, the two linearly independent solutions are

$$e^{-\mu t} \quad \text{and} \quad t e^{-\mu t}.$$

Hence the general displacement is

$$x(t) = (C_1 + C_2 t) e^{-\mu t}, \quad \mu = \omega_0.$$

Applying the initial conditions,

$$x(0) = C_1 = x_0.$$

Differentiation gives

$$\dot{x} = [C_2 - \mu(C_1 + C_2 t)] e^{-\mu t}.$$

Therefore,

$$v_0 = \dot{x}(0) = C_2 - \mu C_1,$$

so

$$C_2 = v_0 + \mu x_0.$$

The displacement is consequently

$$x(t) = [x_0 + (v_0 + \mu x_0)t] e^{-\mu t}, \quad \mu = \omega_0.$$

This motion is non-oscillatory. Critical damping marks the boundary between oscillatory and non-oscillatory motion and gives the fastest non-oscillatory return to equilibrium for the usual release conditions.

The critically damped expression can also be checked as the continuous limit of the underdamped solution. With fixed x_0 and v_0 ,

$$x(t) = e^{-\mu t} \left[x_0 \cos(\omega_d t) + (v_0 + \mu x_0) \frac{\sin(\omega_d t)}{\omega_d} \right].$$

As $\mu \rightarrow \omega_0^-$, one has $\omega_d \rightarrow 0$, and

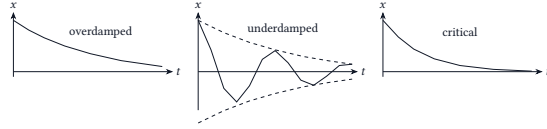
$$\lim_{\omega_d \rightarrow 0} \cos(\omega_d t) = 1, \quad \lim_{\omega_d \rightarrow 0} \frac{\sin(\omega_d t)}{\omega_d} = t.$$

Therefore,

$$x(t) \rightarrow [x_0 + (v_0 + \mu x_0)t] e^{-\mu t},$$

which is exactly the critically damped result.

The characteristic time dependence in the three regimes may be represented schematically as follows; the curves correspond to representative initial conditions and illustrate only the qualitative distinction between monotonic decay and decaying oscillation:



A general physical check follows from the mechanical energy

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2.$$

Its time derivative is

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + m\omega_0^2 x\dot{x} = m\dot{x}(\ddot{x} + \omega_0^2 x).$$

Using the equation of motion,

$$\ddot{x} + \omega_0^2 x = -2\mu\dot{x},$$

gives

$$\frac{dE}{dt} = -2m\mu\dot{x}^2 \leq 0.$$

Thus the damping force continuously removes mechanical energy in all three regimes. For $\mu > 0$, every solution tends to

$$x(t) \rightarrow 0, \quad \dot{x}(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

in agreement with the oscillator ultimately reaching equilibrium.

All exponents are dimensionless because μt , αt , and $\omega_d t$ are dimensionless. In each initial-value formula, the coefficients multiplying the trigonometric or hyperbolic functions have dimensions of displacement:

$$\left[\frac{v_0 + \mu x_0}{\alpha} \right] = \left[\frac{v_0 + \mu x_0}{\omega_d} \right] = L,$$

and

$$[(v_0 + \mu x_0)t] = L.$$

The solutions are therefore dimensionally consistent.

CONCLUSION:

For the signed displacement $x(t)$, with $[\mu] = [\omega_0] = \text{s}^{-1}$, the three damping regimes are:

$\mu > \omega_0$ (overdamped): $x(t) = C_1 e^{[-\mu + \sqrt{\mu^2 - \omega_0^2}]t} + C_2 e^{[-\mu - \sqrt{\mu^2 - \omega_0^2}]t},$
$0 \leq \mu < \omega_0$ (underdamped): $x(t) = e^{-\mu t} \left[C_1 \cos(\sqrt{\omega_0^2 - \mu^2} t) + C_2 \sin(\sqrt{\omega_0^2 - \mu^2} t) \right],$
$\mu = \omega_0$ (critically damped): $x(t) = (C_1 + C_2 t) e^{-\mu t}.$

For specified initial displacement x_0 in metres and initial velocity v_0 in metres per second, these become

$x_{\text{over}}(t) = e^{-\mu t} \left[x_0 \cosh(\alpha t) + \frac{v_0 + \mu x_0}{\alpha} \sinh(\alpha t) \right], \quad \alpha = \sqrt{\mu^2 - \omega_0^2},$
$x_{\text{under}}(t) = e^{-\mu t} \left[x_0 \cos(\omega_d t) + \frac{v_0 + \mu x_0}{\omega_d} \sin(\omega_d t) \right], \quad \omega_d = \sqrt{\omega_0^2 - \mu^2},$
$x_{\text{critical}}(t) = [x_0 + (v_0 + \mu x_0)t] e^{-\mu t}, \quad \mu = \omega_0.$

The underdamped oscillator crosses the equilibrium repeatedly, alternately moving in the positive and negative x -directions, while its amplitude decreases as $e^{-\mu t}$. The overdamped and critically damped oscillators return without sustained oscillation; critical damping is the limiting case separating oscillatory from non-oscillatory behavior and provides the fastest non-oscillatory return under the usual release conditions. In every case with $\mu > 0$, the displacement and velocity tend to zero and the oscillator ultimately comes to rest at its equilibrium position because

$$\frac{dE}{dt} = -2m\mu\dot{x}^2 \leq 0.$$

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Q7. Calculate the group and phase velocities for the wave packet corresponding to a relativistic particle.

INTRODUCTION:

The question requires the phase velocity of the individual matter-wave components and the group velocity of the wave-packet envelope representing a free relativistic particle.

Consider a one-dimensional, narrowly localized wave packet constructed from positive-energy plane waves,

$$\Psi(x, t) = \int A(k) \exp[i\{kx - \omega(k)t\}] dk,$$

where $A(k)$ is sharply peaked about the central wave number k_0 . We choose the $+x$ -axis as the propagation direction; hence $k_0 > 0$ and the particle momentum is $p > 0$. The corresponding results remain valid for signed p and k , with the signs specifying the direction of propagation.

For each plane-wave component, the de Broglie and Planck relations are

$$p = \hbar k, \quad E = \hbar \omega.$$

For a free particle of rest mass m , the relativistic energy-momentum relation is

$$E^2 = p^2 c^2 + m^2 c^4,$$

where the positive-energy branch $E > 0$ is used. The particle velocity in the chosen inertial frame is denoted by v , with

$$p = \gamma mv, \quad E = \gamma mc^2, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

The phase velocity and group velocity are defined respectively by

$$v_{\text{ph}} = \frac{\omega}{k}, \quad v_{\text{g}} = \left. \frac{d\omega}{dk} \right|_{k_0}.$$

The identification of v_{g} with the particle velocity assumes a free particle, a sufficiently narrow distribution in k , and a dispersion relation that varies smoothly over the significant spectral width of the packet.

SOLUTION:

Using $E = \hbar\omega$ and $p = \hbar k$, the relativistic dispersion relation is

$$\hbar^2 \omega^2 = \hbar^2 k^2 c^2 + m^2 c^4.$$

Thus, for positive frequency,

$$\omega(k) = \sqrt{k^2 c^2 + \frac{m^2 c^4}{\hbar^2}}.$$

The phase velocity of a component with wave number $k \neq 0$ is

$$v_{\text{ph}} = \frac{\omega}{k} = \frac{E/\hbar}{p/\hbar} = \frac{E}{p}.$$

From the relativistic expressions for energy and momentum,

$$\frac{E}{p} = \frac{\gamma mc^2}{\gamma mv} = \frac{c^2}{v}.$$

Therefore,

$$v_{\text{ph}} = \frac{E}{p} = \frac{c^2}{v}.$$

For a massive particle, $0 < |v| < c$, so

$$|v_{\text{ph}}| = \frac{c^2}{|v|} > c.$$

This superluminal phase velocity does not represent the transport of a localized particle, energy, or information; it describes the motion of surfaces of constant phase of a monochromatic component.

To find the group velocity, consider two neighboring components of the packet,

$$\Psi_1 = \cos(kx - \omega t)$$

$$\Psi_2 = \cos[(k + \Delta k)x - (\omega + \Delta\omega)t].$$

Their superposition contains a slowly varying envelope whose constant-phase condition is

$$\Delta k x - \Delta\omega t = \text{constant}.$$

Differentiating this condition gives the envelope velocity

$$\frac{dx}{dt} = \frac{\Delta\omega}{\Delta k}.$$

In the narrow-packet limit $\Delta k \rightarrow 0$,

$$v_{\text{g}} = \frac{d\omega}{dk}.$$

Since

$$E = \hbar\omega, \quad p = \hbar k,$$

the chain rule gives

$$v_{\text{g}} = \frac{d(E/\hbar)}{d(p/\hbar)} = \frac{dE}{dp}.$$

Differentiating

$$E = \sqrt{p^2 c^2 + m^2 c^4}$$

with respect to p , one obtains

$$\frac{dE}{dp} = \frac{1}{2} (p^2 c^2 + m^2 c^4)^{-1/2} (2pc^2) = \frac{pc^2}{E}.$$

Hence

$$v_{\text{g}} = \frac{pc^2}{E}.$$

Using $p = \gamma mv$ and $E = \gamma mc^2$,

$$v_{\text{g}} = \frac{(\gamma mv)c^2}{\gamma mc^2} = v.$$

Therefore,

$$v_{\text{g}} = v,$$

so the wave-packet envelope propagates with the classical relativistic velocity of the particle.

The two velocities obey

$$v_{\text{ph}} v_{\text{g}} = \left(\frac{E}{p} \right) \left(\frac{pc^2}{E} \right) = c^2,$$

or

$$v_{\text{ph}} v_{\text{g}} = c^2.$$

Several checks establish the consistency of these results.

The dimensions are

$$\left[\frac{E}{p} \right] = \frac{\text{kg m}^2 \text{s}^{-2}}{\text{kg m s}^{-1}} = \text{m s}^{-1},$$

and

$$\left[\frac{pc^2}{E} \right] = \frac{(\text{kg m s}^{-1})(\text{m}^2 \text{s}^{-2})}{\text{kg m}^2 \text{s}^{-2}} = \text{m s}^{-1},$$

so both expressions have the dimensions of velocity.

In the non-relativistic limit $p \ll mc$,

$$E = \sqrt{m^2 c^4 + p^2 c^2} \approx mc^2 + \frac{p^2}{2m}.$$

Consequently,

$$v_g = \frac{dE}{dp} \approx \frac{p}{m} = v \ll c,$$

whereas, if the full relativistic frequency including the rest-energy term is used,

$$v_{ph} = \frac{E}{p} \approx \frac{mc^2}{p} + \frac{p}{2m} \approx \frac{c^2}{v} \gg c.$$

Thus the envelope moves slowly with the particle, while the carrier phase moves rapidly, as expected.

In the ultra-relativistic limit $p \gg mc$,

$$E \approx pc,$$

and hence

$$v_g \approx c, \quad v_{ph} \approx c.$$

For an exactly massless particle, $E = |p|c$, so both velocities have magnitude c . For a massive particle at $p = 0$, the group velocity is zero, while $v_{ph} = \omega/k$ is not defined at $k = 0$, because there is no spatially propagating carrier phase at that point.

For signed momentum, the positive-energy relation gives

$$v_g = \frac{pc^2}{E}, \quad v_{ph} = \frac{E}{p}.$$

Both velocities therefore have the sign of p , so both reverse direction when the momentum reverses, while their product remains c^2 .

CONCLUSION:

For a free relativistic particle with

$$E = \sqrt{p^2 c^2 + m^2 c^4}, \quad p = \hbar k, \quad E = \hbar \omega,$$

the phase and group velocities are

$$v_{ph} = \frac{\omega}{k} = \frac{E}{p} = \frac{c^2}{v},$$

and

$$v_g = \frac{d\omega}{dk} = \frac{dE}{dp} = \frac{pc^2}{E} = v.$$

Both are measured in m s^{-1} , and their signs give the direction of propagation relative to the chosen $+x$ -axis. They satisfy

$$v_{ph} v_g = c^2.$$

For a massive particle, $|v_g| = |v| < c$ while $|v_{ph}| > c$. The superluminal phase velocity does not violate relativity because it does not transport the particle, energy, or information. Under the free-particle and narrow-wave-packet assumptions, it is the group velocity that gives the physical propagation velocity of the relativistic particle. In the massless or ultra-relativistic limit, both velocity magnitudes approach c .

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Q8. Using Huygens' principle for a plane wave travelling from rarer medium 1 to a denser medium 2, show that $\frac{\sin i}{\sin r} = \frac{v_1}{v_2} = \frac{\mu_2}{\mu_1}$, where i and r are the angles of incidence and refraction, respectively. v_1, μ_1 and v_2, μ_2 are the velocities and refractive indices in media 1 and 2, respectively.

INTRODUCTION:

The required result is Snell's law obtained from the Huygens construction of the refracted wavefront. According to Huygens' principle, every point of a wavefront acts as a source of secondary wavelets, and the forward envelope of these wavelets gives the subsequent wavefront.

Assume that both media are homogeneous, isotropic, transparent, and stationary, and that their refractive indices are positive. The plane interface is taken to be $y = 0$, with medium 1 in $y > 0$ and medium 2 in $y < 0$. The wave travels from medium 1 toward decreasing y . The angles i and r are measured from the normal to the interface. Since medium 2 is optically denser,

$$\mu_2 > \mu_1, \quad v_2 < v_1.$$

At the fixed frequency of the incident wave, the refractive index and phase velocity are related by

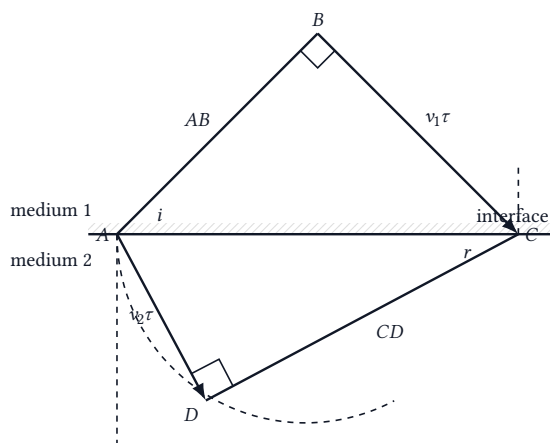
$$\mu_j = \frac{c}{v_j}, \quad j = 1, 2,$$

where c is the speed of light in vacuum.

Let AB be an incident plane wavefront. Point A reaches the interface first. After a time interval τ , point B reaches the interface at C . During the same time, the secondary wavelet emitted from A in medium 2 has radius AD . The tangent CD drawn from C to this wavelet is the refracted wavefront.

SOLUTION:

The Huygens construction is represented schematically below.



The incident ray BC is normal to the incident wavefront AB , so

$$AB \perp BC.$$

The normal to the interface is perpendicular to AC . Hence the angle between the incident ray BC and the interface normal is equal to the angle between AB and AC . Therefore,

$$\angle BAC = i.$$

In the time interval τ , point B of the incident wavefront travels through medium 1 to C . Thus,

$$BC = v_1 \tau.$$

Since $AB \perp BC$, triangle ABC is right-angled at B . Therefore,

$$\sin i = \sin \angle BAC = \frac{BC}{AC} = \frac{v_1 \tau}{AC}.$$

Equivalently,

$$AC = \frac{v_1 \tau}{\sin i}. \quad (1)$$

During precisely the same interval τ , the secondary wavelet originating at A travels through medium 2. Its radius is consequently

$$AD = v_2 \tau.$$

The line CD is tangent to this secondary wavelet at D . A radius drawn to the point of tangency is perpendicular to the tangent, so

$$AD \perp CD.$$

Thus triangle ACD is right-angled at D .

The refracted rays are normal to the refracted wavefront CD . Since the interface normal is perpendicular to AC , the angle between the refracted ray and the interface normal equals the angle between the refracted wavefront CD and the interface AC . Hence,

$$\angle ACD = r.$$

It follows from right triangle ACD that

$$\sin r = \sin \angle ACD = \frac{AD}{AC} = \frac{v_2 \tau}{AC}.$$

Equivalently,

$$AC = \frac{v_2 \tau}{\sin r}. \quad (2)$$

Equating the two expressions for the common distance AC from Eqs. (1) and (2),

$$\frac{v_1 \tau}{\sin i} = \frac{v_2 \tau}{\sin r}.$$

Since $\tau \neq 0$,

$$v_1 \sin r = v_2 \sin i,$$

and hence

$$\boxed{\frac{\sin i}{\sin r} = \frac{v_1}{v_2}}. \quad (3)$$

Using

$$v_1 = \frac{c}{\mu_1}, \quad v_2 = \frac{c}{\mu_2},$$

the velocity ratio becomes

$$\frac{v_1}{v_2} = \frac{c/\mu_1}{c/\mu_2} = \frac{\mu_2}{\mu_1}.$$

Substitution in Eq. (3) gives

$$\boxed{\frac{\sin i}{\sin r} = \frac{v_1}{v_2} = \frac{\mu_2}{\mu_1}}.$$

All three ratios are dimensionless: $\sin i$ and $\sin r$ are dimensionless, v_1/v_2 is a ratio of velocities, and μ_2/μ_1 is a ratio of refractive indices.

As a consistency check, if the two media become optically identical, then

$$\mu_2 = \mu_1 \implies v_2 = v_1 \implies \sin r = \sin i.$$

For $0 \leq i, r < \pi/2$, this gives $r = i$, so there is no bending, as expected. At normal incidence, the limiting form gives $i = r = 0$, and the ray continues along the normal.

For the stated passage from a rarer to a denser medium,

$$\mu_2 > \mu_1 \implies \frac{\mu_2}{\mu_1} > 1.$$

Therefore,

$$\frac{\sin i}{\sin r} > 1 \implies \sin r < \sin i.$$

Since the relevant angles are acute,

$$r < i.$$

Thus the refracted ray bends toward the normal on entering the denser medium.

CONCLUSION:

Huygens' construction gives, for the same propagation time τ ,

$$BC = v_1 \tau, \quad AD = v_2 \tau,$$

and the two right triangles sharing the distance AC give

$$\sin i = \frac{v_1 \tau}{AC}, \quad \sin r = \frac{v_2 \tau}{AC}.$$

Consequently,

$$\boxed{\frac{\sin i}{\sin r} = \frac{v_1}{v_2} = \frac{\mu_2}{\mu_1}}.$$

This result is valid for homogeneous, isotropic media with positive refractive indices and with $v_j = c/\mu_j$ at the common wave frequency. Because medium 2 is denser, $v_2 < v_1$ and $\mu_2 > \mu_1$, so $r < i$: the transmitted ray bends toward the normal.

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Q9. Write equation for damped harmonic oscillations and obtain expression for logarithmic decrement. In a damped harmonic motion, the first amplitude is 10 cm, which reduces to 2 cm after 50 oscillations, each of period 4 seconds. Determine the logarithmic decrement. Also, calculate the number of oscillations in which the amplitude decreases to 25%.

INTRODUCTION:

The problem requires the equation of a freely damped harmonic oscillator, the derivation of its logarithmic decrement, and the use of the observed decay of amplitude to determine both the decrement and the number of oscillations required for a specified reduction.

Let $x(t)$ be the displacement, measured from the equilibrium position along the positive x -axis. For viscous damping, the restoring force and damping force are taken as

$$F_r = -kx, \quad F_d = -c\dot{x},$$

where m is the mass, k is the force constant, and c is the viscous damping coefficient. The damping force is opposite to the instantaneous velocity. No external periodic force is present.

The motion is assumed to be underdamped,

$$c^2 < 4mk,$$

so that the body continues to oscillate while its amplitude decreases exponentially. Define

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad \gamma = \frac{c}{2m}, \quad \omega_d = \sqrt{\omega_0^2 - \gamma^2},$$

where ω_0 is the undamped natural angular frequency, γ is the amplitude-decay constant, and ω_d is the damped angular frequency. The corresponding damped period is

$$T_d = \frac{2\pi}{\omega_d}.$$

The logarithmic decrement δ is defined as the natural logarithm of the ratio of two successive amplitudes measured on the same side of the equilibrium position.

SOLUTION:

1. From Newton's second law,

$$m\ddot{x} = F_r + F_d = -kx - c\dot{x}.$$

Hence the differential equation for free damped harmonic oscillations is

$$\boxed{m\ddot{x} + c\dot{x} + kx = 0}.$$

On division by m ,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0.$$

To solve it, take a trial solution

$$x = e^{st}.$$

Substitution gives the auxiliary equation

$$s^2 + 2\gamma s + \omega_0^2 = 0,$$

whose roots are

$$s = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}.$$

For underdamped motion, $\gamma < \omega_0$, so that

$$s = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2} = -\gamma \pm i\omega_d.$$

Therefore,

$$x(t) = e^{-\gamma t} (C_1 \cos \omega_d t + C_2 \sin \omega_d t).$$

Equivalently,

$$\boxed{x(t) = A_0 e^{-\gamma t} \cos(\omega_d t + \phi)},$$

where A_0 and ϕ are fixed by the initial conditions. Thus the slowly varying amplitude, or envelope, is

$$A(t) = A_0 e^{-\gamma t}.$$

Consider amplitudes A_r and A_{r+1} of two successive maxima on the same side of equilibrium.

These maxima are separated by one damped period T_d . If the first maximum occurs at time t , then

$$A_r = A_0 e^{-\gamma t},$$

and

$$A_{r+1} = A_0 e^{-\gamma(t+T_d)}.$$

Their ratio is therefore

$$\frac{A_r}{A_{r+1}} = \frac{A_0 e^{-\gamma t}}{A_0 e^{-\gamma(t+T_d)}} = e^{\gamma T_d}.$$

By definition, the logarithmic decrement is

$$\delta = \ln \left(\frac{A_r}{A_{r+1}} \right).$$

Consequently,

$$\delta = \gamma T_d.$$

Using $\gamma = c/(2m)$ and $T_d = 2\pi/\omega_d$,

$$\delta = \frac{cT_d}{2m} = \frac{2\pi\gamma}{\omega_d} = \frac{\pi c}{m\sqrt{\omega_0^2 - c^2/(4m^2)}}.$$

More generally, if the amplitudes A_r and A_{r+n} are separated by n complete oscillations, then

$$A_{r+n} = A_r e^{-\gamma n T_d} = A_r e^{-n\delta}.$$

Thus

$$\frac{A_r}{A_{r+n}} = e^{n\delta},$$

and hence

$$\delta = \frac{1}{n} \ln \left(\frac{A_r}{A_{r+n}} \right).$$

The logarithmic decrement is dimensionless, since it is the logarithm of a ratio of two amplitudes. This is also consistent with

$$[\gamma T_d] = s^{-1} s = 1.$$

2. In the given motion,

$$A_1 = 10 \text{ cm}, \quad A_{51} = 2 \text{ cm}, \quad n = 50.$$

The two amplitudes are separated by 50 complete oscillations. Therefore,

$$\delta = \frac{1}{50} \ln \left(\frac{10 \text{ cm}}{2 \text{ cm}} \right).$$

The units cancel within the logarithm:

$$\delta = \frac{1}{50} \ln 5.$$

Using

$$\ln 5 = 1.6094379,$$

one obtains

$$\delta = \frac{1.6094379}{50} = 0.0321888.$$

Hence,

$$\delta \approx 3.22 \times 10^{-2}$$

per oscillation.

Since the period is

$$T_d = 4 \text{ s},$$

the decay constant may also be checked from

$$\gamma = \frac{\delta}{T_d} = \frac{0.0321888}{4 \text{ s}} = 8.0472 \times 10^{-3} \text{ s}^{-1}.$$

Thus the amplitude as a function of time is proportional to

$$A(t) = A_0 e^{-(8.05 \times 10^{-3} \text{ s}^{-1})t}.$$

As a numerical consistency check, the elapsed time in 50 oscillations is

$$t = 50(4 \text{ s}) = 200 \text{ s}.$$

Therefore,

$$\frac{A(200 \text{ s})}{A_0} = e^{-\gamma t} = e^{-(8.0472 \times 10^{-3} \text{ s}^{-1})(200 \text{ s})} = e^{-1.60944} = 0.200.$$

Hence

$$A(200 \text{ s}) = (10 \text{ cm})(0.200) = 2.00 \text{ cm},$$

which reproduces the stated amplitude and verifies the calculation.

3. Let N be the number of oscillations required for the amplitude to become 25% of its initial value. Then

$$A_N = 0.25 A_0.$$

Using

$$A_N = A_0 e^{-N\delta},$$

gives

$$0.25 A_0 = A_0 e^{-N\delta}.$$

After cancelling A_0 ,

$$0.25 = e^{-N\delta}.$$

Taking natural logarithms,

$$\ln(0.25) = -N\delta.$$

Since

$$-\ln(0.25) = \ln 4,$$

the required number is

$$N = \frac{\ln 4}{\delta}.$$

Substituting $\delta = \ln 5/50$,

$$N = \frac{\ln 4}{\ln 5/50} = \frac{50 \ln 4}{\ln 5}.$$

Numerically,

$$N = \frac{50(1.3862944)}{1.6094379} = 43.06.$$

Thus, if a fractional number of periods is permitted,

$$N \simeq 43.1 \text{ oscillations}.$$

The corresponding time is

$$t = NT_d = (43.06)(4 \text{ s}) = 172.2 \text{ s}.$$

If the answer is required in complete oscillations, the amplitude after 43 oscillations is

$$\frac{A_{43}}{A_0} = e^{-43\delta} = e^{-43(0.0321888)} \simeq 0.2506,$$

which is still slightly greater than 25%. After 44 oscillations,

$$\frac{A_{44}}{A_0} = e^{-44\delta} = e^{-44(0.0321888)} \simeq 0.2427,$$

which is below 25%. Hence the amplitude first becomes no more than 25% after

$$44 \text{ complete oscillations}.$$

As a limiting check, when $c \rightarrow 0$, one has $\gamma \rightarrow 0$, so

$$\delta = \gamma T_d \rightarrow 0$$

and $A(t) \rightarrow A_0$. Thus the amplitude becomes constant in the absence of damping, as required by conservation of mechanical energy.

CONCLUSION:

For a freely oscillating mass subject to a linear viscous damping force opposite to its velocity,

$$m\ddot{x} + c\dot{x} + kx = 0.$$

In the underdamped regime $c^2 < 4mk$, its displacement and amplitude are

$$x(t) = A_0 e^{-ct/(2m)} \cos(\omega_d t + \phi)$$

$$A(t) = A_0 e^{-ct/(2m)}$$

where

$$\omega_d = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}.$$

The logarithmic decrement is the dimensionless quantity

$$\delta = \ln \left(\frac{A_r}{A_{r+1}} \right) = \frac{cT_d}{2m} = \frac{2\pi\gamma}{\omega_d},$$

or, for amplitudes separated by n oscillations,

$$\delta = \frac{1}{n} \ln \left(\frac{A_r}{A_{r+n}} \right).$$

For the stated motion,

$$\delta = \frac{\ln 5}{50} \simeq 0.0322 \text{ per oscillation},$$

with the equivalent decay constant

$$\gamma \simeq 8.05 \times 10^{-3} \text{ s}^{-1}.$$

The amplitude falls to exactly 25% of its initial value after

$$N \simeq 43.1 \text{ oscillations}$$

or approximately

$$t \simeq 172 \text{ s}.$$

When only complete oscillations are counted, it first becomes no more than 25% after

$$44 \text{ complete oscillations}.$$

The decreasing positive envelope represents dissipation of mechanical energy by a damping force directed opposite to the velocity; the result is valid for linear viscous damping and underdamped free motion.

IFoS · 2023 · 10 MARKS

Q10. Find an expression for the average energy of a forced oscillator in the steady-state motion.

INTRODUCTION:

Consider a one-dimensional damped harmonic oscillator of mass m , natural angular frequency ω_0 , and linear damping constant b , driven by a sinusoidal force

$$F(t) = F_0 \cos \omega t.$$

Its equation of motion is

$$m\ddot{x} + b\dot{x} + m\omega_0^2 x = F_0 \cos \omega t.$$

It is convenient to define

$$2\beta = \frac{b}{m},$$

so that

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t.$$

Here ω is the driving angular frequency and β is the damping rate. The displacement x is measured from the equilibrium position, with positive x chosen along the positive direction of the applied force.

After the transient motion has decayed, the oscillator executes steady-state motion at the driving frequency. The required average energy is the time average, over one driving period $T = 2\pi/\omega$, of the instantaneous mechanical energy

$$E(t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2.$$

Because damping continually dissipates energy, $E(t)$ is not constant; the applied force supplies, on average, exactly the energy dissipated during steady-state motion.

SOLUTION:

The steady-state displacement may be written as

$$x(t) = A \cos(\omega t - \delta),$$

where A is the displacement amplitude and δ is the phase lag relative to the applied force. Differentiating,

$$\dot{x}(t) = -A\omega \sin(\omega t - \delta)$$

$$\ddot{x}(t) = -A\omega^2 \cos(\omega t - \delta).$$

To obtain A , it is useful to represent the steady-state motion by

$$x(t) = a \cos \omega t + c \sin \omega t.$$

Then

$$\dot{x} = -a\omega \sin \omega t + c\omega \cos \omega t,$$

and

$$\ddot{x} = -a\omega^2 \cos \omega t - c\omega^2 \sin \omega t.$$

Substitution into the equation of motion gives

$$[(\omega_0^2 - \omega^2)a + 2\beta\omega c] \cos \omega t + [(\omega_0^2 - \omega^2)c - 2\beta\omega a] \sin \omega t = \frac{F_0}{m} \cos \omega t.$$

Equating the coefficients of $\cos \omega t$ and $\sin \omega t$,

$$(\omega_0^2 - \omega^2)a + 2\beta\omega c = \frac{F_0}{m},$$

$$(\omega_0^2 - \omega^2)c - 2\beta\omega a = 0.$$

Squaring and adding the corresponding amplitude relations yields

$$[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2] (a^2 + c^2) = \left(\frac{F_0}{m}\right)^2.$$

Since $A^2 = a^2 + c^2$,

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}.$$

Equivalently, in terms of $b = 2m\beta$,

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}}.$$

The phase lag is determined by

$$\tan \delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2},$$

with the quadrant selected so that $0 \leq \delta \leq \pi$ for positive damping.

The instantaneous kinetic energy is

$$K(t) = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\omega^2 \sin^2(\omega t - \delta),$$

while the instantaneous elastic potential energy is

$$U(t) = \frac{1}{2}m\omega_0^2 x^2 = \frac{1}{2}m\omega_0^2 A^2 \cos^2(\omega t - \delta).$$

The average of a quantity $Q(t)$ over one driving period is

$$\langle Q \rangle = \frac{1}{T} \int_{t_0}^{t_0+T} Q(t) dt.$$

Since

$$\langle \sin^2(\omega t - \delta) \rangle = \langle \cos^2(\omega t - \delta) \rangle = \frac{1}{2},$$

the average kinetic and potential energies are

$$\langle K \rangle = \frac{1}{4}mA^2\omega^2,$$

and

$$\langle U \rangle = \frac{1}{4}mA^2\omega_0^2.$$

Therefore, the average total mechanical energy is

$$\langle E \rangle = \langle K \rangle + \langle U \rangle = \frac{1}{4}mA^2(\omega^2 + \omega_0^2).$$

Substituting the steady-state amplitude,

$$\langle E \rangle = \frac{F_0^2}{4m} \frac{\omega^2 + \omega_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}.$$

In terms of the damping constant b , the same result is

$$\langle E \rangle = \frac{F_0^2 m (\omega^2 + \omega_0^2)}{4 [m^2 (\omega_0^2 - \omega^2)^2 + b^2 \omega^2]}.$$

The result passes the dimensional check. Since

$$[A] = \text{m}, \quad [m(\omega^2 + \omega_0^2)A^2] = \text{kg s}^{-2} \text{m}^2 = \text{J},$$

the expression has the dimensions of energy.

Several physical limits provide useful checks. In the quasistatic limit $\omega \rightarrow 0$,

$$A \rightarrow \frac{F_0}{m\omega_0^2}, \quad \langle K \rangle \rightarrow 0,$$

and hence

$$\langle E \rangle \rightarrow \frac{F_0^2}{4m\omega_0^2}.$$

This is the time-averaged elastic energy associated with the slowly varying displacement

$$x(t) \simeq \frac{F_0}{m\omega_0^2} \cos \omega t.$$

At $\omega = \omega_0$,

$$A = \frac{F_0}{2m\beta\omega_0} = \frac{F_0}{b\omega_0},$$

and the mean kinetic and potential energies become equal:

$$\langle K \rangle = \langle U \rangle.$$

Consequently,

$$\langle E \rangle_{\omega=\omega_0} = \frac{F_0^2}{8m\beta^2} = \frac{mF_0^2}{2b^2}.$$

This finite resonant value requires nonzero damping. In the idealized limit $\beta \rightarrow 0$ at $\omega = \omega_0$, the assumed bounded steady-state motion ceases to exist, and the above steady-state energy diverges, consistently indicating resonant growth.

As a steady-state energy-balance check, the average rate of dissipation is

$$\langle P_{\text{diss}} \rangle = b \langle \dot{x}^2 \rangle = \frac{1}{2} b A^2 \omega^2.$$

The average power delivered by the force is

$$\langle P_{\text{in}} \rangle = \langle F(t) \dot{x}(t) \rangle = \frac{1}{2} F_0 A \omega \sin \delta.$$

Using

$$F_0 \sin \delta = b A \omega,$$

one obtains

$$\langle P_{\text{in}} \rangle = \langle P_{\text{diss}} \rangle,$$

as required for a periodic steady state.

CONCLUSION:

For a linearly damped oscillator driven by $F_0 \cos \omega t$, the steady-state displacement is

$$x(t) = A \cos(\omega t - \delta),$$

where

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

$$\tan \delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

$$2\beta = \frac{b}{m}.$$

Its average kinetic energy, potential energy, and total mechanical energy are, respectively,

$$\langle K \rangle = \frac{1}{4} m A^2 \omega^2, \quad \langle U \rangle = \frac{1}{4} m A^2 \omega_0^2,$$

and

$$\langle E \rangle = \frac{1}{4} m A^2 (\omega^2 + \omega_0^2) = \frac{F_0^2}{4m} \frac{\omega^2 + \omega_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}.$$

The energy is measured in joules and is nonnegative. The result applies after all transient motion has decayed, for sinusoidal forcing and linear viscous damping. At $\omega = \omega_0$, the mean kinetic and potential energies are equal, while nonzero damping keeps the steady-state energy finite. The oscillator does not conserve mechanical energy instantaneously; instead, in steady state, the average power supplied by the driving force equals the average power dissipated by damping.



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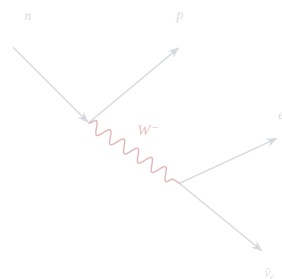
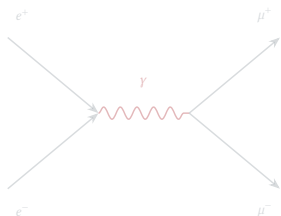
$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$E_n = -\frac{13.6}{n^2} \text{ eV}$$

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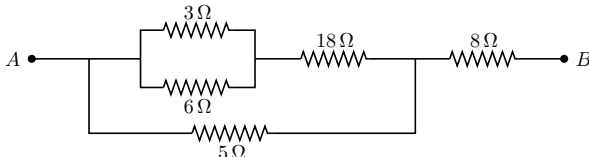
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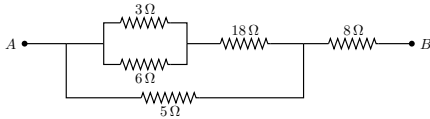
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2	A charge of $-3 \cdot 30 \mu\text{C}$ is fixed at a point. From a horizontal distance of $0 \cdot 0455 \text{ m}$, a charged particle of mass $7 \cdot 35 \times 10^{-3} \text{ kg}$ and charge $-7 \cdot 45 \mu\text{C}$ is fired with an initial velocity $62 \cdot 5 \text{ m/s}$ directly towards the fixed charge. How far does this charge travel before its speed becomes zero?	10	2026	CSE	4
3	Discuss the origin of hysteresis in ferromagnetic materials on the basis of domain theory. Explain how domain wall motion and pinning lead to energy loss during cyclic magnetization.	20	2026	CSE	5
4	A wire of length 2 m is perpendicular to the X - Y plane. It is moved with velocity $\vec{V} = 2\hat{i} + 3\hat{j} + \hat{k} \text{ m/s}$ through a region of uniform magnetic induction $\vec{B} = \hat{i} + 2\hat{j} \text{ Wb/m}^2$. Compute the potential difference induced between the ends of the wire.	15	2026	CSE	6
5	A very long solenoid of radius a , with n turns per unit length, carries a current I_s . Coaxial with the solenoid, at radius $b \gg a$, is a circular ring of wire with resistance R . When the current in the solenoid is gradually reduced, a current I_r is induced in the ring. Calculate I_r in terms of $\frac{dI_s}{dt}$. Also, calculate the power dissipated through Joule effect and the electric field $\vec{E}$ near the solenoid.	15	2026	CSE	7
6	The space between the plates of a parallel-plate capacitor is filled with a dielectric material whose susceptibility varies linearly from 0 at the bottom plate ($x = 0$) to 1 at the top plate ($x = d$). The capacitor is connected to a battery of voltage V . Calculate all the bound charges and check that the total charge is zero. Assume that the battery is connected in such a way that the electric field points along the x -direction while the free charge density, σ_f , is positive at the bottom plate and negative at the top plate.	20	2026	CSE	8
7	Consider a point charge of 5 nC placed at a distance of 1 m from a perfect conducting plane ($z = 0$) of infinite extent. Find the electric field at a point $(2, 2, 0) \text{ m}$ and show that it is normal to the plane.	10	2025	CSE	10
8	A rectangular coil consists of 50 closely wrapped turns and has dimensions of $0 \cdot 5 \text{ m} \times 0 \cdot 4 \text{ m}$. It carries a current of $1 \cdot 5 \text{ A}$. If a uniform magnetic field $B = 0 \cdot 1 \text{ T}$ is applied such that the direction of the magnetic field makes an angle of 60° with respect to the plane of the coil, what is the torque exerted on the coil by the magnetic field?	10	2025	CSE	11
9	State and explain Kirchhoff's current law and Kirchhoff's voltage law. Derive these laws from the principles of charge conservation and energy conservation.	10	2025	CSE	12
10	A parallel plate capacitor having circular plates of radius 10 cm is being charged. If the electric field at any instant within the capacitor changes at the rate $5 \cdot 0 \text{ V m}^{-1} \text{ s}^{-1}$, calculate the magnetic intensity $ \vec{H} $ inside the capacitor.	10	2025	CSE	15

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Q1. Calculate the effective resistance of the following combination of resistances as shown in the figure and determine the voltage drop across each resistance when a potential difference of 120 volts is applied between points A and B :

**INTRODUCTION:**

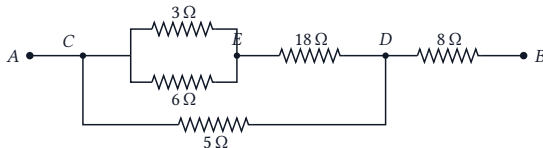
The circuit is to be reduced by identifying resistors connected between the same pair of nodes and then applying the series and parallel combination rules. All resistances are assumed ideal, and the connecting wires have negligible resistance. Let

$$V_A - V_B = 120 \text{ V},$$

so conventional current is taken from A toward B . Denote by C the node immediately after A , by D the junction before the 8Ω resistor, and by E the junction between the 3Ω - 6Ω parallel pair and the 18Ω resistor. Ohm's law and Kirchhoff's current and voltage laws will be used.

SOLUTION:

The circuit topology is



The 3Ω and 6Ω resistors are connected between the same nodes C and E , and hence are in parallel:

$$R_{CE} = \frac{(3 \Omega)(6 \Omega)}{3 \Omega + 6 \Omega} = \frac{18}{9} \Omega = 2 \Omega.$$

This 2Ω equivalent is in series with the 18Ω resistor along the upper path from C to D . Therefore,

$$R_{\text{upper}} = 2 \Omega + 18 \Omega = 20 \Omega.$$

The upper 20Ω path and the lower 5Ω path have the same end nodes C and D , so they are in parallel:

$$R_{CD} = \frac{(20 \Omega)(5 \Omega)}{20 \Omega + 5 \Omega} = \frac{100}{25} \Omega = 4 \Omega.$$

Finally, R_{CD} is in series with the 8Ω resistor:

$$R_{AB} = R_{CD} + 8 \Omega = 4 \Omega + 8 \Omega = 12 \Omega.$$

The total current supplied between A and B is consequently

$$I = \frac{V_A - V_B}{R_{AB}} = \frac{120 \text{ V}}{12 \Omega} = 10 \text{ A}.$$

Since the 8Ω resistor is in series with the entire C - D network, it carries 10 A . Its voltage drop from D toward B is

$$V_8 = I(8 \Omega) = (10 \text{ A})(8 \Omega) = 80 \text{ V}.$$

Kirchhoff's voltage law then gives the voltage across the parallel network between C and D :

$$V_{CD} = 120 \text{ V} - 80 \text{ V} = 40 \text{ V}.$$

The 5Ω resistor is directly connected between C and D , so

$$V_5 = V_{CD} = 40 \text{ V}, \quad I_5 = \frac{40 \text{ V}}{5 \Omega} = 8 \text{ A}.$$

The complete upper path also has a 40 V drop. Its current is

$$I_{\text{upper}} = \frac{40 \text{ V}}{20 \Omega} = 2 \text{ A}.$$

Thus, the drop across the 18Ω resistor is

$$V_{18} = I_{\text{upper}}(18 \Omega) = (2 \text{ A})(18 \Omega) = 36 \text{ V}.$$

The remaining voltage across the 3Ω - 6Ω parallel pair is

$$V_{CE} = V_{CD} - V_{18} = 40 \text{ V} - 36 \text{ V} = 4 \text{ V}.$$

Since both resistors are connected between C and E ,

$$V_3 = V_6 = 4 \text{ V}.$$

Their respective currents are

$$I_3 = \frac{4 \text{ V}}{3 \Omega} = \frac{4}{3} \text{ A}, \quad I_6 = \frac{4 \text{ V}}{6 \Omega} = \frac{2}{3} \text{ A}.$$

These add to

$$I_3 + I_6 = \frac{4}{3} \text{ A} + \frac{2}{3} \text{ A} = 2 \text{ A},$$

which is the current entering the pair. At node C ,

$$I_5 + I_{\text{upper}} = 8 \text{ A} + 2 \text{ A} = 10 \text{ A},$$

in agreement with the total circuit current. The voltage drops along the upper path also satisfy

$$V_{3||6} + V_{18} = 4 \text{ V} + 36 \text{ V} = 40 \text{ V},$$

and the complete A -to- B drop is

$$40 \text{ V} + 80 \text{ V} = 120 \text{ V}.$$

CONCLUSION:

The effective resistance between A and B is

$$R_{AB} = 12 \Omega,$$

and the total conventional current is

$$I = 10 \text{ A}$$

from A toward B . The voltage decreases in the direction of this current, with the individual drops

$$V_{3\Omega} = 4 \text{ V}, \quad V_{6\Omega} = 4 \text{ V}, \quad V_{18\Omega} = 36 \text{ V}, \quad V_{5\Omega} = 40 \text{ V}, \quad V_{8\Omega} = 80 \text{ V}.$$

These results apply for ideal ohmic resistors and negligible connecting-wire resistance. The equal 4 V drops across the 3Ω and 6Ω resistors follow from their parallel connection, while the 5Ω resistor has the same 40 V drop as the entire alternative upper branch.

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Q2. A charge of $-3.30 \mu\text{C}$ is fixed at a point. From a horizontal distance of 0.0455 m , a charged particle of mass $7.35 \times 10^{-3} \text{ kg}$ and charge $-7.45 \mu\text{C}$ is fired with an initial velocity 62.5 m/s directly towards the fixed charge. How far does this charge travel before its speed becomes zero?

INTRODUCTION:

The required quantity is the distance travelled by the moving charge before it reaches its point of closest approach and momentarily comes to rest. Since both charges are negative, their electrostatic interaction is repulsive. The particle is therefore decelerated as it moves towards the fixed charge.

Let the fixed charge be at $x = 0$, and let the moving particle initially be at $x = r_i > 0$. The positive x -direction is taken from the fixed charge towards the particle. Thus the initial velocity is directed along $-x$, whereas the electrostatic force is along $+x$.

Write

$$Q = -3.30 \times 10^{-6} \text{ C}, \quad q = -7.45 \times 10^{-6} \text{ C},$$

$$m = 7.35 \times 10^{-3} \text{ kg}$$

$$r_i = 0.0455 \text{ m}$$

$$v_i = 62.5 \text{ m s}^{-1}.$$

The fixed charge is assumed to remain stationary, both charges are treated as point charges, and forces other than the electrostatic force are neglected. Since the electrostatic force is conservative, mechanical energy is conserved.

SOLUTION:

For two point charges separated by a distance r , the electrostatic potential energy is

$$U(r) = \frac{kQq}{r}, \quad k = 8.99 \times 10^9 \text{ N m}^2 \text{ C}^{-2}.$$

Here

$$Qq = (-3.30 \times 10^{-6})(-7.45 \times 10^{-6}) = 2.4585 \times 10^{-11} \text{ C}^2 > 0,$$

so $U(r)$ increases as the separation decreases. The particle's initial kinetic energy is converted into electrostatic potential energy.

Let r_t denote the separation at the turning point, where the speed becomes zero. Conservation of mechanical energy gives

$$\frac{1}{2}mv_i^2 + \frac{kQq}{r_i} = 0 + \frac{kQq}{r_t}.$$

Hence

$$\frac{1}{r_t} = \frac{1}{r_i} + \frac{mv_i^2}{2kQq}.$$

The initial kinetic energy is

$$\begin{aligned} K_i &= \frac{1}{2}(7.35 \times 10^{-3})(62.5)^2 \\ &= \frac{1}{2}(7.35 \times 10^{-3})(3906.25) \\ &= 14.36 \text{ J}. \end{aligned}$$

Also,

$$\begin{aligned} kQq &= (8.99 \times 10^9)(2.4585 \times 10^{-11}) \\ &= 2.21 \times 10^{-1} \text{ J m}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{r_t} &= \frac{1}{0.0455} + \frac{14.36}{2.21 \times 10^{-1}} \\ &= 21.98 \text{ m}^{-1} + 64.98 \text{ m}^{-1} \\ &= 86.96 \text{ m}^{-1}, \end{aligned}$$

and hence

$$r_t = \frac{1}{86.96} = 1.15 \times 10^{-2} \text{ m}.$$

The distance travelled towards the fixed charge is the decrease in separation:

$$\begin{aligned} s &= r_i - r_t \\ &= 0.0455 - 0.0115 \\ &= 0.0340 \text{ m}. \end{aligned}$$

Since $r_t < r_i$, the result correctly describes motion towards the fixed charge. At r_t , the positive electrostatic potential energy has increased by exactly the initial kinetic energy, so the particle momentarily stops and then reverses direction under the repulsive force.

CONCLUSION:

The moving charge travels

$$s = 3.40 \times 10^{-2} \text{ m}$$

or

$$s = 3.40 \text{ cm}$$

towards the fixed negative charge before its speed becomes zero. Its minimum separation from the fixed charge is

$$r_t = 1.15 \times 10^{-2} \text{ m} = 1.15 \text{ cm}.$$

This result applies under the point-charge approximation with the fixed charge stationary and all non-electrostatic forces neglected.

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Q3. Discuss the origin of hysteresis in ferromagnetic materials on the basis of domain theory. Explain how domain wall motion and pinning lead to energy loss during cyclic magnetization.

INTRODUCTION:

A ferromagnetic specimen below its Curie temperature is divided into magnetic domains. Within each domain, the atomic magnetic moments are aligned approximately parallel by the exchange interaction, so that the domain possesses a spontaneous magnetization M_s . Adjacent domains are separated by transition regions called domain walls, within which the magnetization rotates continuously from the direction in one domain to that in the next.

The domain structure minimizes the total free energy, which contains exchange, magnetocrystalline anisotropy, magnetostatic, magnetoelastic, and domain-wall contributions. In zero applied field, differently oriented domains generally arrange themselves so that the net macroscopic magnetization is small, even though each domain is strongly magnetized.

Let the applied magnetic field H and the measured magnetization M be positive along a fixed axis. In SI units,

$$B = \mu_0(H + M).$$

The hysteresis loop arises because the domain configuration at a given H depends on the previous magnetic history. The principal microscopic cause of this history dependence is the irreversible displacement

of domain walls through an inhomogeneous material containing defects, internal stresses, grain boundaries, inclusions, dislocations, and spatial variations of magnetic anisotropy.

SOLUTION:

When a field H is applied, the magnetic contribution to the free-energy density of a domain whose magnetization makes an angle θ with the field is

$$u_H = -\mu_0 \mathbf{M} \cdot \mathbf{H} = -\mu_0 M_s H \cos \theta.$$

Thus, domains whose magnetization lies more nearly parallel to the field have lower energy. Magnetization can then increase by two principal processes:

1. Domain walls move so that domains favourably oriented with respect to the field grow at the expense of unfavourably oriented domains.
2. At larger fields, the magnetization within the remaining domains rotates towards the field direction. Eventually the specimen approaches magnetic saturation.

Consider a wall separating two domains whose magnetization components along the applied field are M_1 and M_2 . If the wall moves through a distance dx , converting an elemental volume $A dx$ from domain 1 to domain 2, the change in magnetic energy is

$$dU_H = -\mu_0 H(M_2 - M_1)A dx.$$

Hence the field exerts an effective pressure on the wall,

$$p_H = \mu_0 H(M_2 - M_1).$$

For a 180° wall separating domains with components $-M_s$ and $+M_s$,

$$p_H = 2\mu_0 M_s H.$$

The wall therefore tends to move in the direction that enlarges the domain magnetized parallel to the field.

In an ideal, perfectly homogeneous crystal, a wall could move comparatively freely and much of the response would be reversible. A real ferromagnet, however, has a non-uniform internal energy landscape. A defect can lower the wall energy when the wall intersects it, or an internal stress can locally alter the magnetoelastic and anisotropy energies. Such regions act as pinning centres.

Let $U_p(x)$ denote the position-dependent pinning energy of a wall. The effective wall energy may be represented schematically by

$$U_{\text{eff}}(x; H) = U_p(x) - \mu_0 H(M_2 - M_1)Ax.$$

The applied field tilts this energy landscape. For a small field, a pinned wall may deform or move slightly within a local energy minimum. Such motion is approximately reversible: when the field is removed, the wall returns nearly to its original position.

As H increases, the field pressure increases. At a critical depinning field H_p , the local minimum ceases to hold the wall. In force form, the depinning condition is approximately

$$\mu_0 H_p (M_2 - M_1) A \simeq \max \left| \frac{dU_p}{dx} \right|.$$

The wall then jumps abruptly to another stable position. This is an irreversible change because, after the jump, merely restoring the field to its previous value does not restore the former domain configuration. The wall can return only when the field is reversed sufficiently to overcome a generally different pinning barrier. A succession of such discontinuous jumps produces Barkhausen noise.

Starting from an initially demagnetized state, increasing H causes favourable domains to grow by wall motion. At higher fields, domain rotation becomes important, and sufficiently large H produces approximate saturation. When the field is subsequently reduced to zero, many walls do not retrace their earlier paths because they become trapped at pinning centres. Consequently, a finite magnetization remains:

$$M(H = 0) = M_r,$$

where M_r is the remanent magnetization.

A reverse field is required to move the pinned walls and reverse enough domains to make the net magnetization vanish. The magnitude of the reverse field for which $M = 0$ is the coercive field H_c . Further increase of the reverse field produces negative saturation. Repeating the process completes a closed hysteresis loop.

The work supplied by the external source per unit volume in a quasistatic magnetic cycle is

$$W_{\text{cycle}} = \oint H dB.$$

Using $B = \mu_0(H + M)$,

$$W_{\text{cycle}} = \mu_0 \oint H dH + \mu_0 \oint H dM.$$

Since H returns to its initial value,

$$\oint H dH = \oint d\left(\frac{H^2}{2}\right) = 0,$$

and therefore

$$W_{\text{loss}} = \mu_0 \oint H dM.$$

Thus the energy dissipated per unit volume in one complete cycle equals the area enclosed by the B - H loop,

$$W_{\text{loss}} = \oint H dB,$$

or equivalently μ_0 times the area enclosed by the M - H loop. Its unit is

$$[H dB] = \frac{\text{A}}{\text{m}} \text{T} = \frac{\text{J}}{\text{m}^3},$$

as required for an energy density.

Microscopically, the applied field stores energy while a domain wall is held against a pinning barrier. Upon depinning, the wall accelerates and jumps to another metastable position. The released energy is transferred to lattice vibrations, electronic excitations, microscopic eddy currents in conducting specimens, and ultimately heat. Since the reverse path encounters barriers in a different sequence and direction, the energy released during forward motion is not fully recovered. The magnetization consequently lags behind the applied field, producing hysteresis and a positive energy loss per cycle.

The character of the hysteresis loop is therefore closely related to the strength of pinning. Soft ferromagnetic materials contain relatively mobile domain walls and have small H_c , narrow loops, and low hysteresis loss. Hard ferromagnetic materials possess strong pinning and large anisotropy; they consequently have large coercivity, broad loops, and large remanence, making them suitable for permanent magnets.

CONCLUSION:

Ferromagnetic hysteresis originates from the existence of multiple metastable domain configurations separated by energy barriers. An applied field lowers the energy of domains aligned with it, causing domain-wall motion and, at larger fields, domain rotation. Structural defects, internal stresses, grain boundaries, inclusions, and anisotropy variations pin the walls. Walls depin only after a finite threshold field is reached and then move irreversibly to new positions, so the domain pattern on decreasing or reversing the field differs from that formed while increasing it.

This irreversible wall motion accounts for remanent magnetization M_r , coercive field H_c , and the lag of M behind H . The magnetic energy dissipated per unit volume per cycle is

$$W_{\text{loss}} = \oint H dB = \mu_0 \oint H dM,$$

equal to the enclosed B - H loop area, or to μ_0 times the enclosed M - H loop area. The lost magnetic work ultimately appears mainly as heat; stronger wall pinning generally produces a wider hysteresis loop and a larger cyclic energy loss.

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Q4. A wire of length 2 m is perpendicular to the X - Y plane. It is moved with velocity $\vec{V} = 2\hat{i} + 3\hat{j} + \hat{k}$ m/s through a region of uniform magnetic induction $\vec{B} = \hat{i} + 2\hat{j}$ Wb/m². Compute the potential difference induced between the ends of the wire.

INTRODUCTION:

The wire is perpendicular to the X - Y plane and is therefore parallel to the z -axis. Let its lower end be A , its upper end in the $+\hat{k}$ direction be C , and

$$\vec{L}_{AC} = 2\hat{k} \text{ m.}$$

The calculation is made in the laboratory frame, in which the magnetic field is uniform and the wire translates rigidly with velocity $\vec{V}$. The wire is assumed to be open-circuited. The magnetic Lorentz force $q(\vec{V} \times \vec{B})$ separates charges until an electrostatic potential difference is established between its ends.

With the stated orientation, the required terminal potential difference is

$$V_C - V_A = \int_A^C (\vec{V} \times \vec{B}) \cdot d\vec{l}.$$

SOLUTION:

The motional-force vector per unit charge is

$$\begin{aligned} \vec{V} \times \vec{B} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 1 & 2 & 0 \end{vmatrix} \\ &= (3 \cdot 0 - 1 \cdot 2)\hat{i} - (2 \cdot 0 - 1 \cdot 1)\hat{j} + (2 \cdot 2 - 3 \cdot 1)\hat{k} \\ &= -2\hat{i} + \hat{j} + \hat{k} \text{ V m}^{-1}. \end{aligned}$$

Only the component of $\vec{V} \times \vec{B}$ parallel to the wire contributes to the terminal potential difference. Since $d\vec{l} = \hat{k} dz$,

$$\begin{aligned} V_C - V_A &= \int_0^2 (-2\hat{i} + \hat{j} + \hat{k}) \cdot \hat{k} dz \\ &= \int_0^2 1 dz \\ &= 2 \text{ V.} \end{aligned}$$

The $+\hat{k}$ component of $\vec{V} \times \vec{B}$ drives positive charges toward C . Thus the end in the $+\hat{k}$ direction is at the higher potential.

The result also follows from the scalar triple product

$$V_C - V_A = (\vec{V} \times \vec{B}) \cdot \vec{L}_{AC} = 2 \text{ V.}$$

This shows that the velocity component parallel to the wire does not contribute directly; the nonzero voltage arises from the transverse velocity and magnetic-field components whose cross product has a component along the wire.

CONCLUSION:

The magnitude of the induced potential difference between the ends of the wire is

$$\Delta V = 2 \text{ V.}$$

For the convention $\vec{L}_{AC} = 2\hat{k}$ m,

$$V_C - V_A = +2 \text{ V,}$$

so the end lying in the $+\hat{k}$ direction is 2 V above the end lying in the $-\hat{k}$ direction.

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Q5. A very long solenoid of radius a , with n turns per unit length, carries a current I_s . Coaxial with the solenoid, at radius $b \gg a$, is a circular ring of wire with resistance R . When the current in the solenoid is gradually reduced, a current I_r is induced in the ring. Calculate I_r in terms of $\frac{dI_s}{dt}$. Also, calculate the power dissipated through Joule effect and the electric field $\vec{E}$ near the solenoid.

INTRODUCTION:

For an ideal, very long solenoid, the magnetic field is uniform inside and negligible outside:

$$\vec{B} = \begin{cases} \mu_0 n I_s \hat{z}, & r < a, \\ 0, & r > a. \end{cases}$$

Although the ring has radius $b \gg a$, the magnetic flux linked with it comes only from the cross-sectional area πa^2 of the solenoid.

Choose $+\hat{z}$ along the magnetic field produced by positive I_s , and choose $+\hat{\phi}$ counterclockwise as viewed from $+\hat{z}$. Faraday's law,

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt},$$

determines both the induced emf and the circulating electric field. The current is assumed to vary quasistatically, displacement-current and radiation effects are neglected, and the ring's self-inductance is neglected compared with its resistance; equivalently, the variation time scale is much longer than its L/R time.

SOLUTION:

The magnetic flux through the ring is

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = (\mu_0 n I_s)(\pi a^2).$$

Therefore, the induced emf around the ring is

$$\mathcal{E}_r = -\frac{d\Phi_B}{dt} = -\mu_0 n \pi a^2 \frac{dI_s}{dt}.$$

Using Ohm's law for the ring,

$$I_r = \frac{\mathcal{E}_r}{R},$$

gives

$$I_r = -\frac{\mu_0 n \pi a^2}{R} \frac{dI_s}{dt}.$$

Here $I_r > 0$ denotes current in the $+\hat{\phi}$ direction. Since the solenoid current is being reduced,

$$\frac{dI_s}{dt} < 0,$$

and hence $I_r > 0$: the induced current is counterclockwise when viewed from $+\hat{z}$. Its magnetic field points along $+\hat{z}$, opposing the decrease of the solenoid's flux, as required by Lenz's law.

The Joule power dissipated in the ring is

$$P_J = I_r^2 R.$$

Substitution yields

$$P_J = \frac{\mu_0^2 n^2 \pi^2 a^4}{R} \left(\frac{dI_s}{dt} \right)^2.$$

The square makes the dissipated power positive irrespective of whether I_s is increasing or decreasing.

To determine the electric field, symmetry requires it to be azimuthal:

$$\vec{E}(r) = E_\phi(r) \hat{\phi}.$$

For a coaxial circular path of radius r ,

$$\oint_C \vec{E} \cdot d\vec{l} = 2\pi r E_\phi(r).$$

For $r < a$, the enclosed magnetic flux is

$$\Phi_B(r) = \mu_0 n I_s \pi r^2.$$

Thus,

$$2\pi r E_\phi(r) = -\mu_0 n \pi r^2 \frac{dI_s}{dt},$$

so that

$$\vec{E}(r) = -\frac{\mu_0 n r}{2} \frac{dI_s}{dt} \hat{\phi}, \quad r < a.$$

For $r \geq a$, only the solenoid cross-section contributes to the flux:

$$\Phi_B(r) = \mu_0 n I_s \pi a^2.$$

Consequently,

$$2\pi r E_\phi(r) = -\mu_0 n \pi a^2 \frac{dI_s}{dt},$$

and hence

$$\vec{E}(r) = -\frac{\mu_0 n a^2}{2r} \frac{dI_s}{dt} \hat{\phi}, \quad r \geq a.$$

The two expressions agree at $r = a$, where

$$\vec{E}(a) = -\frac{\mu_0 n a}{2} \frac{dI_s}{dt} \hat{\phi}.$$

At the ring, $r = b$,

$$\vec{E}(b) = -\frac{\mu_0 n a^2}{2b} \frac{dI_s}{dt} \hat{\phi},$$

and its circulation is

$$2\pi b E_\phi(b) = -\mu_0 n \pi a^2 \frac{dI_s}{dt} = \mathcal{E}_r,$$

which consistently reproduces the induced emf.

CONCLUSION:

With positive azimuthal direction chosen counterclockwise as viewed from the $+\hat{z}$ end of the solenoid,

$$I_r = -\frac{\mu_0 n \pi a^2}{R} \frac{dI_s}{dt} \quad (\text{A})$$

and

$$P_J = \frac{\mu_0^2 n^2 \pi^2 a^4}{R} \left(\frac{dI_s}{dt} \right)^2 \quad (\text{W}).$$

The induced electric field is

$$\vec{E}(r) = \begin{cases} -\frac{\mu_0 n r}{2} \frac{dI_s}{dt} \hat{\phi}, & r < a, \\ -\frac{\mu_0 n a^2}{2r} \frac{dI_s}{dt} \hat{\phi}, & r \geq a, \end{cases} \quad (\text{V m}^{-1}).$$

In particular,

$$\vec{E}(b) = -\frac{\mu_0 n a^2}{2b} \frac{dI_s}{dt} \hat{\phi}.$$

For $\frac{dI_s}{dt} < 0$, both $\vec{E}$ and I_r point in $+\hat{\phi}$, and the induced magnetic field opposes the reduction of the original flux. These results apply to an ideal long solenoid and to sufficiently slow current variation that the ring's self-inductance may be neglected.

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Q6. The space between the plates of a parallel-plate capacitor is filled with a dielectric material whose susceptibility varies linearly from 0 at the bottom plate ($x = 0$) to 1 at the top plate ($x = d$). The capacitor is connected to a battery of voltage V . Calculate all the bound charges and check that the total charge is zero. Assume that the battery is connected in such a way that the electric field points along the x -direction while the free charge density, σ_f , is positive at the bottom plate and negative at the top plate.

INTRODUCTION:

The dielectric susceptibility is

$$\chi_e(x) = \frac{x}{d}, \quad 0 \leq x \leq d,$$

so that the position-dependent permittivity is

$$\epsilon(x) = \epsilon_0 \left[1 + \chi_e(x) \right] = \epsilon_0 \left(1 + \frac{x}{d} \right).$$

The dielectric is assumed to be linear, isotropic and locally responding, and fringing fields are neglected. There is no free volume charge within the dielectric. We choose $+\hat{\mathbf{x}}$ from the bottom plate to the top plate. Consistent with the stated orientation,

$$\mathbf{E} = E(x)\hat{\mathbf{x}}, \quad V \equiv \phi(0) - \phi(d) = \int_0^d E(x) dx > 0.$$

The free surface charges on the conducting plates are $+\sigma_f$ at $x = 0$ and $-\sigma_f$ at $x = d$.

The governing relations are

$$\nabla \cdot \mathbf{D} = \rho_f, \quad \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon(x)\mathbf{E},$$

together with

$$\rho_b = -\nabla \cdot \mathbf{P}, \quad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is the outward unit normal from the dielectric.

SOLUTION:

Since $\rho_f = 0$ throughout $0 < x < d$,

$$\frac{dD_x}{dx} = 0.$$

Thus D_x is constant. At the positively charged bottom conductor, the normal component of $\mathbf{D}$ entering the dielectric equals σ_f , and hence

$$\mathbf{D} = \sigma_f \hat{\mathbf{x}}.$$

It follows that the electric field is nonuniform:

$$\mathbf{E}(x) = \frac{\mathbf{D}}{\epsilon(x)} = \frac{\sigma_f}{\epsilon_0(1+x/d)} \hat{\mathbf{x}} = \frac{\sigma_f d}{\epsilon_0(d+x)} \hat{\mathbf{x}}.$$

The applied voltage determines σ_f :

$$V = \int_0^d E(x) dx = \frac{\sigma_f d}{\epsilon_0} \int_0^d \frac{dx}{d+x}.$$

Therefore,

$$V = \frac{\sigma_f d}{\epsilon_0} [\ln(d+x)]_0^d = \frac{\sigma_f d}{\epsilon_0} \ln 2,$$

and

$$\sigma_f = \frac{\epsilon_0 V}{d \ln 2}.$$

The polarization is

$$\mathbf{P}(x) = \epsilon_0 \chi_e(x) \mathbf{E}(x).$$

Substitution gives

$$\mathbf{P}(x) = \epsilon_0 \frac{x}{d} \frac{\sigma_f d}{\epsilon_0(d+x)} \hat{\mathbf{x}} = \frac{\sigma_f x}{d+x} \hat{\mathbf{x}}.$$

The bound volume-charge density is consequently

$$\rho_b(x) = -\nabla \cdot \mathbf{P} = -\frac{dP_x}{dx}.$$

Since

$$\frac{d}{dx} \left(\frac{x}{d+x} \right) = \frac{d}{(d+x)^2},$$

one obtains

$$\rho_b(x) = -\frac{\sigma_f d}{(d+x)^2} = -\frac{\epsilon_0 V}{\ln 2} \frac{1}{(d+x)^2} \quad 0 < x < d.$$

Thus the continuously distributed bound charge is negative everywhere in the dielectric. This is consistent with P_x increasing toward the top plate: a positive polarization gradient produces negative bound volume charge.

At the bottom surface $x = 0$, the outward normal from the dielectric is

$$\hat{\mathbf{n}}_0 = -\hat{\mathbf{x}}.$$

Because $\chi_e(0) = 0$, the polarization there vanishes:

$$\mathbf{P}(0) = 0.$$

Hence

$$\sigma_b(0) = \mathbf{P}(0) \cdot (-\hat{\mathbf{x}}) = 0.$$

At the top surface $x = d$, the outward normal is

$$\hat{\mathbf{n}}_d = +\hat{\mathbf{x}},$$

and

$$\mathbf{P}(d) = \frac{\sigma_f d}{d+d} \hat{\mathbf{x}} = \frac{\sigma_f}{2} \hat{\mathbf{x}}.$$

Therefore,

$$\sigma_b(d) = \mathbf{P}(d) \cdot \hat{\mathbf{x}} = \frac{\sigma_f}{2} = \frac{\epsilon_0 V}{2d \ln 2}.$$

Any lateral surface has its normal perpendicular to $\mathbf{P}$, so its bound surface-charge density is zero within the parallel-plate approximation.

To check the total bound charge, let the plate area be A . The bound volume charge is

$$Q_b^{(\text{vol})} = A \int_0^d \rho_b(x) dx = -A\sigma_f d \int_0^d \frac{dx}{(d+x)^2}.$$

Using

$$\int_0^d \frac{dx}{(d+x)^2} = \left[-\frac{1}{d+x} \right]_0^d = \frac{1}{2d},$$

we find

$$Q_b^{(\text{vol})} = -\frac{A\sigma_f}{2}.$$

The total bound surface charge is

$$Q_b^{(\text{surf})} = A[\sigma_b(0) + \sigma_b(d)] = \frac{A\sigma_f}{2}.$$

Consequently,

$$Q_b^{(\text{total})} = Q_b^{(\text{vol})} + Q_b^{(\text{surf})} = -\frac{A\sigma_f}{2} + \frac{A\sigma_f}{2} = 0.$$

The cancellation is required because polarization merely redistributes the initially neutral dielectric's positive and negative charges.

CONCLUSION:

For the field directed from the positively charged bottom plate toward the negatively charged top plate,

$$\mathbf{E}(x) = \frac{V}{\ln 2} \frac{1}{d+x} \hat{\mathbf{x}}, \quad \mathbf{P}(x) = \frac{\epsilon_0 V}{d \ln 2} \frac{x}{d+x} \hat{\mathbf{x}},$$

and the free surface-charge magnitude is

$$\sigma_f = \frac{\epsilon_0 V}{d \ln 2} \quad \text{in } \text{C m}^{-2}.$$

All bound charges in the one-dimensional parallel-plate approximation are

$$\rho_b(x) = -\frac{\epsilon_0 V}{\ln 2} \frac{1}{(d+x)^2} \quad \text{in } \text{C m}^{-3}, \quad 0 < x < d,$$

$$\sigma_b(0) = 0, \quad \sigma_b(d) = \frac{\epsilon_0 V}{2d \ln 2} \quad \text{in } \text{C m}^{-2},$$

with zero bound charge on the lateral surfaces. The negative volume charge,

$$Q_b^{(\text{vol})} = -\frac{A\epsilon_0 V}{2d \ln 2},$$

is exactly balanced by the positive bound charge on the top dielectric surface,

$$Q_b^{(\text{surf})} = +\frac{A\epsilon_0 V}{2d \ln 2}.$$

Hence the net bound charge is

$$Q_b^{(\text{total})} = 0,$$

as required for an initially neutral dielectric under the stated linear-dielectric and negligible-fringing-field assumptions.

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Q7. Consider a point charge of 5 nC placed at a distance of 1 m from a perfect conducting plane ($z = 0$) of infinite extent. Find the electric field at a point (2, 2, 0) m and show that it is normal to the plane.

INTRODUCTION:

Take the conducting region to be $z \leq 0$ and the physical space containing the charge to be $z > 0$. Since only the perpendicular distance of the charge from the plane is specified, choose the origin at its perpendicular projection on the plane. Thus,

$$q = 5 \times 10^{-9} \text{ C}, \quad \mathbf{r}_q = (0, 0, a), \quad a = 1 \text{ m}.$$

The observation point is

$$P : \quad \mathbf{r}_P = (2, 2, 0) \text{ m}.$$

The electric field requested at the conductor surface means the limiting field immediately outside it, $z = 0^+$; the field inside a perfect conductor is zero. The $+\hat{\mathbf{z}}$ direction is taken from the conductor into the exterior region.

The governing principle is the method of images. The grounded perfect conducting plane is replaced, for $z > 0$, by an image charge

$$q' = -q$$

at the mirror position

$$\mathbf{r}_{q'} = (0, 0, -a).$$

This construction makes the plane $z = 0$ an equipotential of zero potential. By the uniqueness theorem,

the resulting field in $z > 0$ is the physical electrostatic field.

SOLUTION:

For a source charge Q at $\mathbf{r}_0$, Coulomb's law gives

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q(\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3}.$$

Let

$$k = \frac{1}{4\pi\epsilon_0} = 8.98755 \times 10^9 \text{ N m}^2 \text{ C}^{-2}.$$

The displacement from the real charge to P is

$$\mathbf{R}_q = \mathbf{r}_P - \mathbf{r}_q = (2, 2, -1) \text{ m},$$

whereas that from the image charge to P is

$$\mathbf{R}_{q'} = \mathbf{r}_P - \mathbf{r}_{q'} = (2, 2, 1) \text{ m}.$$

Their magnitudes are equal:

$$R_q = R_{q'} = \sqrt{2^2 + 2^2 + 1^2} = 3 \text{ m}.$$

Therefore, the field at P , approached from $z > 0$, is

$$\begin{aligned} \mathbf{E}(P) &= k \left[\frac{q\mathbf{R}_q}{R_q^3} + \frac{q'\mathbf{R}_{q'}}{R_{q'}^3} \right] \\ &= \frac{kq}{(3 \text{ m})^3} [(2, 2, -1) - (2, 2, 1)] \text{ m} \\ &= \frac{kq}{27 \text{ m}^3} (0, 0, -2) \text{ m} \\ &= -\frac{2kq}{27 \text{ m}^2} \hat{\mathbf{z}}. \end{aligned}$$

Substituting the numerical values,

$$\begin{aligned} \mathbf{E}(P) &= -\frac{2(8.98755 \times 10^9 \text{ N m}^2 \text{ C}^{-2})(5 \times 10^{-9} \text{ C})}{27 \text{ m}^2} \hat{\mathbf{z}} \\ &= -3.33 \hat{\mathbf{z}} \text{ N C}^{-1}. \end{aligned}$$

The cancellation of tangential components is explicit:

$$E_x = \frac{kq}{27}(2 - 2) = 0, \quad E_y = \frac{kq}{27}(2 - 2) = 0,$$

while

$$E_z = \frac{kq}{27}(-1 - 1) = -\frac{2kq}{27} \neq 0.$$

Since the plane $z = 0$ has normal vector $\hat{\mathbf{z}}$, the field

$$\mathbf{E}(P) = E_z \hat{\mathbf{z}}$$

has no component parallel to the plane and is therefore strictly normal to it.

More generally, at any surface point $(x, y, 0)$,

$$\mathbf{E}(x, y, 0^+) = -\frac{2kqa}{(x^2 + y^2 + a^2)^{3/2}} \hat{\mathbf{z}},$$

which confirms that the tangential cancellation is a consequence of the mirror symmetry and is not peculiar to P .

The dimensions are consistent:

$$\frac{kq}{R^2} = \frac{(\text{N m}^2 \text{ C}^{-2})(\text{C})}{\text{m}^2} = \text{N C}^{-1}.$$

As a physical check, for $x^2 + y^2 \gg a^2$, the surface field decreases as

$$|\mathbf{E}| \sim \frac{2kqa}{(x^2 + y^2)^{3/2}},$$

rather than as the field of an isolated charge; this is consistent with the real charge and its opposite image forming a dipole-like configuration at large distance. The negative sign also has the expected meaning: for $q > 0$, negative charge is induced on the conductor, so the exterior field points toward the conducting plane.

CONCLUSION:

Immediately outside the conducting plane at $(2, 2, 0) \text{ m}$,

$$\mathbf{E}(2, 2, 0^+) = -3.33 \hat{\mathbf{z}} \text{ N C}^{-1}.$$

Thus,

$$E_x = E_y = 0, \quad E_z = -3.33 \text{ N C}^{-1},$$

so the electric field is normal to the plane and directed into the conductor, along $-\hat{\mathbf{z}}$. This result assumes that the charge is at $(0, 0, 1) \text{ m}$, with the origin chosen at its perpendicular projection on the infinite conducting plane.

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Q8. A rectangular coil consists of 50 closely wrapped turns and has dimensions of $0.5 \text{ m} \times 0.4 \text{ m}$. It carries a current of 1.5 A . If a uniform magnetic field $B = 0.1 \text{ T}$ is applied such that the direction of the magnetic field makes an angle of 60° with respect to the plane of the coil, what is the torque exerted on the coil by the magnetic field?

INTRODUCTION:

The demand is to determine both the magnitude and orientation of the magnetic torque on the current-carrying coil. A planar coil of N turns, area A , and current I has magnetic dipole moment

$$\boldsymbol{\mu} = NIA \hat{\mathbf{n}},$$

where $\hat{n}$ is the unit normal to the plane of the coil, with its sense fixed by the right-hand rule relative to the current. In a uniform magnetic field $\mathbf{B}$, the torque is

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}, \quad \tau = \mu B \sin \theta,$$

where θ is the angle between the coil normal and $\mathbf{B}$.

The field is stated to make an angle $\alpha = 60^\circ$ with the plane of the coil. Therefore, its angle with the normal is

$$\theta = 90^\circ - \alpha = 30^\circ.$$

The coil is assumed rigid, planar, and sufficiently small that the applied field is uniform over all its turns.

SOLUTION:

The area enclosed by each turn is

$$A = (0.5 \text{ m})(0.4 \text{ m}) = 0.20 \text{ m}^2.$$

Hence, the magnitude of the magnetic dipole moment of the 50-turn coil is

$$\begin{aligned} \mu &= NIA \\ &= (50)(1.5 \text{ A})(0.20 \text{ m}^2) \\ &= 15 \text{ A m}^2. \end{aligned}$$

Using $\theta = 30^\circ$, the torque magnitude is

$$\begin{aligned} \tau &= \mu B \sin \theta \\ &= (15 \text{ A m}^2)(0.1 \text{ T}) \sin 30^\circ \\ &= (15)(0.1) \left(\frac{1}{2} \right) \text{ A m}^2 \text{ T} \\ &= 0.75 \text{ N m}. \end{aligned}$$

Equivalently, since the given angle α is measured from the plane,

$$\tau = NIAB \cos \alpha = (50)(1.5)(0.20)(0.1) \cos 60^\circ = 0.75 \text{ N m}.$$

The dimensional check is

$$[\tau] = [\mu B] = (\text{A m}^2)(\text{T}) = \text{N m},$$

which is the correct dimension of torque.

The torque vector is

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}.$$

It is perpendicular to the plane containing $\boldsymbol{\mu}$ and $\mathbf{B}$, with its sense given by the right-hand rule. Since the direction of current around the coil is not specified, the absolute clockwise or anticlockwise sense cannot be assigned. Physically, the torque rotates the coil so as to align $\boldsymbol{\mu}$ with $\mathbf{B}$, thereby making the plane of the coil perpendicular to the magnetic field.

As a limiting check, if the magnetic field were normal to the coil, then $\theta = 0$ and $\tau = 0$, as expected for an already aligned magnetic moment. If the field were parallel to the plane, then $\theta = 90^\circ$ and the torque would attain its maximum value $NIAB$. The calculated value is one-half of this maximum because $\sin 30^\circ = 1/2$.

CONCLUSION:

The torque exerted on the coil is

$$\tau = 0.75 \text{ N m}.$$

Its direction is along $\boldsymbol{\mu} \times \mathbf{B}$, perpendicular to the plane containing the coil normal and the magnetic field. It tends to rotate the coil until its magnetic moment is parallel to $\mathbf{B}$, or equivalently until the plane of the coil is perpendicular to the field. The precise clockwise or anticlockwise sense depends on the unspecified direction of current. This result applies for a rigid planar coil placed in a uniform magnetic field.

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Q9. State and explain Kirchhoff's current law and Kirchhoff's voltage law. Derive these laws from the principles of charge conservation and energy conservation.

INTRODUCTION:

Kirchhoff's laws relate currents and voltages in lumped electrical networks. Kirchhoff's current law follows from conservation of electric charge, while Kirchhoff's voltage law follows from conservation of energy.

We make the lumped-circuit and quasi-static assumptions: circuit elements and junctions have negligible spatial extent compared with the relevant electromagnetic wavelength, connecting wires are ideal unless their resistance is represented explicitly, and no unrepresented time-varying magnetic flux links a circuit loop.

For the current law, consider a small fixed control volume enclosing a junction. A branch current is taken as positive when directed out of this volume. For the voltage law, choose an arbitrary direction of traversal around a closed loop and define

$$\Delta V_k = V_{\text{out},k} - V_{\text{in},k}$$

as the potential change across the k -th element in that direction. Thus, traversal from the negative to the positive terminal of a source gives $+\mathcal{E}$, whereas traversal through a resistor in the direction of current gives $-IR$.

SOLUTION:

1. Kirchhoff's current law

The law states that the algebraic sum of all currents meeting at a junction is zero:

$$\sum_k I_k = 0.$$

With all currents entering the junction taken as positive, the equivalent form is

$$\sum I_{\text{in}} = \sum I_{\text{out}}.$$

The local statement of charge conservation is the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

where ρ is the volume charge density and $\mathbf{J}$ is the current density.

Integrating over a control volume $\mathcal{V}$ enclosing the junction gives

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\tau + \int_{\partial \mathcal{V}} \nabla \cdot \mathbf{J} d\tau = 0.$$

Using the divergence theorem,

$$\frac{d}{dt} \int_{\mathcal{V}} \rho d\tau + \oint_{\partial \mathcal{V}} \mathbf{J} \cdot d\mathbf{a} = 0.$$

If

$$Q_j = \int_{\mathcal{V}} \rho d\tau$$

is the charge stored at the junction, and outward current is positive, then

$$\oint_{\partial \mathcal{V}} \mathbf{J} \cdot d\mathbf{a} = \sum_k I_k.$$

Consequently,

$$\frac{dQ_j}{dt} + \sum_k I_k = 0.$$

In an ordinary lumped circuit, a junction does not continuously accumulate charge. Therefore,

$$\frac{dQ_j}{dt} = 0,$$

and hence

$$\sum_k I_k = 0.$$

If currents entering the junction are assigned positive signs and currents leaving it negative signs, this becomes

$$I_1 + I_2 + \dots - I_3 - I_4 - \dots = 0,$$

or

$$\sum I_{\text{in}} = \sum I_{\text{out}}.$$

The result has a direct physical meaning. If the total incoming current exceeded the outgoing current, positive charge would accumulate at the junction; if the outgoing current exceeded the incoming current, the junction would lose charge. In steady operation neither accumulation can continue, so the two totals must be equal.

The dimensions are consistent:

$$\left[\frac{dQ}{dt} \right] = \text{C s}^{-1} = \text{A} = [I].$$

For a node capable of storing charge, such as a capacitive node, the more general equation

$$\sum_k I_k = -\frac{dQ_j}{dt}$$

must be retained.

2. Kirchhoff's voltage law

The law states that the algebraic sum of all potential changes around any closed circuit loop is zero:

$$\sum_k \Delta V_k = 0.$$

Equivalently, the sum of voltage rises equals the sum of voltage drops:

$$\sum \mathcal{E} = \sum V_{\text{drop}}.$$

To derive the law, let a small positive test charge q make one complete traversal of a closed loop. Because it returns to its initial point, its initial and final electrical states are the same. In steady circuit operation it also has no net change in kinetic energy after a complete circuit. Conservation of energy therefore requires the algebraic sum of the energy changes supplied to and removed from the charge to vanish:

$$\sum_k \Delta U_k = 0.$$

Since voltage is energy transferred per unit charge,

$$\Delta V_k = \frac{\Delta U_k}{q},$$

we obtain

$$q \sum_k \Delta V_k = 0.$$

For $q \neq 0$,

$$\sum_k \Delta V_k = 0.$$

The same result may be written explicitly using node potentials. Suppose a loop passes successively through nodes with potentials

$$V_1, V_2, \dots, V_n, V_1.$$

The total potential change is

$$(V_2 - V_1) + (V_3 - V_2) + \dots + (V_1 - V_n).$$

Every intermediate potential occurs once with a positive sign and once with a negative sign, so the expression telescopes:

$$(V_2 - V_1) + (V_3 - V_2) + \dots + (V_1 - V_n) = 0.$$

Thus,

$$\oint dV = 0.$$

For example, consider a loop containing a source of emf $\mathcal{E}$ and resistors $R_1, \dots, R_n$ carrying the same current I . Traverse the source from its negative to its positive terminal and each resistor in the direction of current. The source produces the voltage rise

$$+\mathcal{E},$$

while Ohm's law gives the voltage change across each resistor as

$$-IR_k.$$

Kirchhoff's voltage law then gives

$$\mathcal{E} - IR_1 - IR_2 - \dots - IR_n = 0,$$

so that

$$\mathcal{E} = I \sum_{k=1}^n R_k.$$

Multiplying by the charge q shows the energy balance directly:

$$q\mathcal{E} = \sum_{k=1}^n qIR_k.$$

Multiplying instead by I gives the corresponding power balance:

$$I\mathcal{E} = \sum_{k=1}^n I^2 R_k.$$

Thus the power supplied by the source equals the power dissipated in the resistors, as required by conservation of energy.

The dimensional check is

$$[\mathcal{E}] = [IR] = A \Omega = V = J C^{-1}.$$

The sign convention also gives the correct limiting case: if all resistances tend to zero while a finite ideal emf remains, the equation would require an unbounded current. This signals that a real loop must contain resistance, internal source resistance, inductance, or another element that limits the current.

If a time-dependent magnetic flux Φ_B links the loop, the electric field is not conservative. Faraday's law gives

$$\oint \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt}.$$

Therefore the simple form $\sum_k \Delta V_k = 0$ applies only when the linked magnetic flux is constant or when the induced emf

$$\mathcal{E}_{\text{ind}} = -\frac{d\Phi_B}{dt}$$

is included explicitly as a circuit emf. With every source, including induction, represented consistently, the generalized energy balance is again

$$\sum \mathcal{E} - \sum V_{\text{drop}} = 0.$$

CONCLUSION:

Kirchhoff's current law is

$$\sum_k I_k = 0 \quad \text{or} \quad \sum I_{\text{in}} = \sum I_{\text{out}},$$

where current is measured in amperes and its sign is fixed by the declared direction at the junction. It is the lumped-circuit consequence of charge conservation and is valid when the junction has no net charge accumulation. More generally,

$$\frac{dQ_j}{dt} + \sum_k I_k = 0.$$

Kirchhoff's voltage law is

$$\sum_k \Delta V_k = 0 \quad \text{or} \quad \sum \mathcal{E} = \sum V_{\text{drop}},$$

where voltage is measured in volts and each sign is determined by the chosen direction of loop traversal. It expresses conservation of energy: a charge returning to its starting point has zero net energy change, so energy supplied per unit charge equals energy absorbed or dissipated per unit charge.

The ordinary voltage law requires a lumped, quasi-static circuit with no omitted time-varying magnetic flux. If such a flux is present, its induced emf

$$\mathcal{E}_{\text{ind}} = -\frac{d\Phi_B}{dt}$$

must be included. Hence Kirchhoff's laws express, respectively, the impossibility of sustained charge creation or loss at a junction and the impossibility of net energy gain or loss in one complete circuit traversal.

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Q10. A parallel plate capacitor having circular plates of radius 10 cm is being charged. If the electric field at any instant within the capacitor changes at the rate $5.0 \text{ V m}^{-1} \text{ s}^{-1}$, calculate the magnetic intensity $|\vec{H}|$ inside the capacitor.

INTRODUCTION:

The changing electric field between the plates produces a displacement current and hence a magnetic field. The appropriate governing principle is the Ampere–Maxwell law,

$$\oint_C \vec{H} \cdot d\vec{l} = \frac{d}{dt} \int_S \vec{D} \cdot d\vec{S}, \quad \vec{D} = \epsilon_0 \vec{E}$$

for air or vacuum between the plates.

Let the circular plates have radius

$$a = 10 \text{ cm} = 0.10 \text{ m},$$

and let $r \leq a$ denote the perpendicular distance from their common axis. Fringing is neglected, so the electric field is taken as uniform and perpendicular to the plates. By cylindrical symmetry, $\vec{H}$ is tangential to a circle of radius r and has constant magnitude on that circle. Its direction is determined by the right-hand rule with respect to the displacement-current direction. The stated rate is interpreted as

$$\left| \frac{\partial E}{\partial t} \right| = 5.0 \text{ V m}^{-1} \text{ s}^{-1}.$$

SOLUTION:

Choose a circular Amperian loop of radius $r \leq a$, concentric with the plates. Since $\vec{H}$ is tangential and uniform along this loop,

$$\oint_C \vec{H} \cdot d\vec{l} = H(r)(2\pi r).$$

The electric flux through the surface bounded by the loop is

$$\Phi_E = E(\pi r^2).$$

Therefore, the displacement current enclosed by the loop is

$$I_d(r) = \epsilon_0 \frac{d\Phi_E}{dt} = \epsilon_0 \pi r^2 \frac{\partial E}{\partial t}.$$

Applying the Ampere–Maxwell law,

$$H(r)(2\pi r) = \epsilon_0 \pi r^2 \left| \frac{\partial E}{\partial t} \right|.$$

Thus, at any radial position inside the plate area,

$$H(r) = \frac{\epsilon_0 r}{2} \left| \frac{\partial E}{\partial t} \right|, \quad 0 \leq r \leq a.$$

Substituting the given rate,

$$H(r) = \frac{(8.854 \times 10^{-12} \text{ F m}^{-1})r}{2} (5.0 \text{ V m}^{-1} \text{ s}^{-1}),$$

or

$$H(r) = 2.21 \times 10^{-11} r \text{ A m}^{-1}$$

when r is expressed in metres.

Since the question specifies the plate radius but no separate observation radius, the intended value is the maximum field inside the capacitor, attained near the plate rim at $r = a = 0.10 \text{ m}$:

$$\begin{aligned} H(a) &= \frac{(8.854 \times 10^{-12})(0.10)(5.0)}{2} \text{ A m}^{-1} \\ &= 2.2135 \times 10^{-12} \text{ A m}^{-1} \\ &\approx \boxed{2.2 \times 10^{-12} \text{ A m}^{-1}}. \end{aligned}$$

The dimensional check is

$$\left[\epsilon_0 r \frac{\partial E}{\partial t} \right] = \frac{\text{C}}{\text{V m}} \text{ m} \frac{\text{V}}{\text{m s}} = \frac{\text{C}}{\text{m s}} = \frac{\text{A}}{\text{m}},$$

which is the correct dimension of magnetic intensity. Also,

$$\lim_{r \rightarrow 0} H(r) = 0,$$

as required by cylindrical symmetry, and $H(r)$ increases linearly from the axis to the rim.

CONCLUSION:

The magnetic intensity at a distance r from the common axis, within the circular plate area, is

$$|\vec{H}(r)| = \frac{\epsilon_0 r}{2} \left| \frac{\partial E}{\partial t} \right| = 2.21 \times 10^{-11} r \text{ A m}^{-1}, \quad 0 \leq r \leq 0.10 \text{ m}.$$

At the rim, where the magnitude is maximum,

$$|\vec{H}|_{\text{max}} = 2.2 \times 10^{-12} \text{ A m}^{-1}.$$

The field circulates azimuthally around the capacitor axis, with its sense given by the right-hand rule relative to the direction of the increasing electric field. This result assumes a uniform electric field and neglects fringing near the plate edge.



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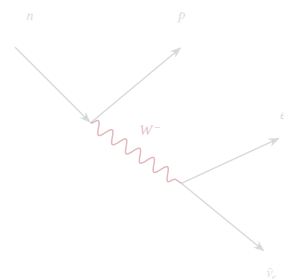
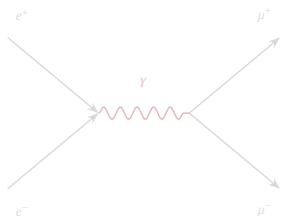
$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$E_n = -\frac{13.6}{n^2} \text{ eV}$$

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$$\partial_\mu j^\mu = 0$$

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Sample Preview extbar 10 Solved Questions

Question Index

No.	Question	Marks	Year	Exam	Page
1	What are the limitations of the first law of thermodynamics? One mole of a gas, assumed to be perfect, at 0°C is heated at constant pressure till its volume is twice its initial value. Calculate the amount of heat absorbed. Given, $C_v = 20 \cdot 9 \text{ J} \cdot \text{mol}^{-1} \cdot \text{K}^{-1}$ and $R = 8 \cdot 3 \text{ J} \cdot \text{mol}^{-1} \cdot \text{K}^{-1}$.	10	2026	CSE	3
2	Two solids A and B have Debye temperatures 200 K and 300 K respectively. At $T = 20 \text{ K}$, compare their specific heats.	10	2026	CSE	4
3	Calculate the pressure at which water will boil at 150°C , given that the change in specific volume when 1 gram of water is converted into steam is 1676 cm^3 . Given, latent heat of vaporization for steam = 540 cal per gram , $J = 4 \cdot 2 \times 10^7 \text{ ergs/cal}$ and one atmospheric pressure = 10^6 dynes/cm^2 .	15	2026	CSE	4
4	Why does the specific heat of solids depend on material at low temperatures but become independent of material at high temperatures?	5	2026	CSE	5
5	A reversible heat engine operates with three reservoirs at 300 K , 400 K and 1200 K . It absorbs 1200 kJ energy as heat from the reservoir at 1200 K and delivers 400 kJ work. Determine the heat interactions with the other two reservoirs.	10	2025	CSE	7
6	A ternary system consists of three components (A , B and C) in equilibrium with two phases. Determine the number of degrees of freedom using the Gibb's phase rule and discuss the effect of pressure and temperature variations on the phase equilibrium.	15	2025	CSE	8
7	Discuss briefly the considerations which led Van der Waals to modify the gas equation. What are the critical constants of a gas? Calculate the values of these constants in terms of the constants of the Van der Waals equation.	15	2025	CSE	10
8	Explain why, at equilibrium, the chemical potential of a component must be the same in all coexisting phases. Derive the equilibrium condition for a binary liquid-vapour system in terms of chemical potential.	15	2025	CSE	12
9	For silver, the specific heat at constant pressure in the range of 50 K to 100 K is given by : $C_p = 0 \cdot 076 T - 0 \cdot 00026 T^2 - 0 \cdot 15 \text{ cal mol}^{-1} \text{ deg}^{-1}$ If 2 moles of silver are heated from 50 K to 100 K , calculate the change in entropy.	8	2025	IFoS	14
10	One gram of water (1 cm^3) becomes 1671 cm^3 of steam when boiled at constant pressure of 1 atm ($1 \cdot 013 \times 10^5 \text{ Pa}$). The heat of vapourisation at this pressure is $L = 2 \cdot 256 \times 10^6 \text{ J/kg}$. Calculate : (i) The work done by the water when it vapourizes, and; (ii) Increase in its internal energy.	10	2025	IFoS	15

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Q1. What are the limitations of the first law of thermodynamics? One mole of a gas, assumed to be perfect, at 0 °C is heated at constant pressure till its volume is twice its initial value. Calculate the amount of heat absorbed. Given, $C_v = 20 \cdot 9 \text{ J} \cdot \text{mol}^{-1} \cdot \text{K}^{-1}$ and $R = 8 \cdot 3 \text{ J} \cdot \text{mol}^{-1} \cdot \text{K}^{-1}$.

INTRODUCTION:

The first law of thermodynamics is the principle of conservation of energy applied to a thermodynamic system. With heat supplied to the gas taken as positive and work done by the gas taken as positive, it is written as

$$\Delta U = Q - W, \quad \text{or} \quad Q = \Delta U + W.$$

It relates heat, work and the change in internal energy, but by itself does not specify whether a process can occur spontaneously or in which direction it will proceed.

For the calculation, the gas is assumed to remain ideal and in quasistatic equilibrium while being heated at constant pressure. Let T_1, V_1 and T_2, V_2 denote its initial and final temperatures and volumes. The given middle dots in the numerical data are interpreted as decimal points:

$$C_v = 20.9 \text{ J mol}^{-1} \text{K}^{-1}, \quad R = 8.3 \text{ J mol}^{-1} \text{K}^{-1}.$$

The initial absolute temperature is

$$T_1 = 273.15 \text{ K},$$

and the final volume satisfies $V_2 = 2V_1$.

SOLUTION:

The limitations of the first law are as follows:

1. It gives only an energy balance; it does not indicate the natural direction of a process. For example, it does not explain why heat flows spontaneously from a hotter body to a colder body and not conversely.
2. It provides no criterion for spontaneity or thermodynamic equilibrium. Processes having the same energy balance may differ in whether they can actually occur.
3. It does not distinguish between reversible and irreversible processes and gives no measure of the degradation or quality of energy.
4. It places no restriction, by itself, on the conversion of heat into work. Although it forbids a perpetual-motion machine of the first kind, which would create energy, it cannot rule out a

hypothetical cyclic engine that converts all the heat received from a single reservoir into work. That restriction follows from the second law.

5. It does not determine the maximum possible efficiency of a heat engine or the coefficient of performance of a refrigerator.
6. It does not introduce entropy and hence cannot account for entropy production in irreversible processes.
7. It gives no information about the rate at which a permitted process occurs.

For the numerical part, the ideal-gas equation is

$$PV = nRT.$$

Since P and n remain constant,

$$\frac{V_2}{V_1} = \frac{T_2}{T_1}.$$

Using $V_2 = 2V_1$,

$$\frac{T_2}{T_1} = 2, \quad T_2 = 2T_1 = 2(273.15 \text{ K}) = 546.30 \text{ K}.$$

Therefore,

$$\Delta T = T_2 - T_1 = 546.30 - 273.15 = 273.15 \text{ K}.$$

For one mole of an ideal gas,

$$\Delta U = nC_v \Delta T.$$

The work done during constant-pressure expansion is

$$W = P(V_2 - V_1).$$

Using $PV = nRT$ at the initial and final states,

$$P(V_2 - V_1) = nR(T_2 - T_1) = nR\Delta T.$$

Hence, from the first law,

$$Q = \Delta U + W = nC_v \Delta T + nR\Delta T = n(C_v + R)\Delta T.$$

Thus the constant-pressure molar heat capacity is

$$C_p = C_v + R = 20.9 + 8.3 = 29.2 \text{ J mol}^{-1} \text{K}^{-1}.$$

The heat absorbed is therefore

$$\begin{aligned} Q &= nC_p \Delta T \\ &= (1.00 \text{ mol})(29.2 \text{ J mol}^{-1} \text{K}^{-1})(273.15 \text{ K}) \\ &= 7975.98 \text{ J} \\ &\approx 7.98 \times 10^3 \text{ J}. \end{aligned}$$

For physical consistency, both $\Delta U = nC_v \Delta T$ and $W = nR\Delta T$ are positive because the gas is heated and expands. Consequently $Q = \Delta U + W > 0$, as required for heat absorbed by the gas.

CONCLUSION:

The first law expresses conservation of energy but does not determine the direction or spontaneity of a process, distinguish reversible from irreversible change, introduce entropy, set the maximum efficiency of heat engines, rule out complete cyclic conversion of heat from a single reservoir into work, or predict process rates.

At constant pressure, doubling the volume of the ideal gas doubles its absolute temperature. Thus

$$\begin{aligned} T_2 &= 546.30 \text{ K} \\ \Delta T &= 273.15 \text{ K} \\ C_p &= 29.2 \text{ J mol}^{-1} \text{ K}^{-1}. \end{aligned}$$

Therefore, the amount of heat absorbed by one mole of the gas is

$$Q \approx 7.98 \times 10^3 \text{ J} = 7.98 \text{ kJ}.$$

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Q2. Two solids *A* and *B* have Debye temperatures 200 K and 300 K respectively. At $T = 20 \text{ K}$, compare their specific heats.

INTRODUCTION:

The temperature is much smaller than either Debye temperature:

$$\frac{T}{\Theta_{D,A}} = \frac{20}{200} = 0.10, \quad \frac{T}{\Theta_{D,B}} = \frac{20}{300} = 0.0667.$$

Hence the low-temperature Debye T^3 law is applicable. The comparison is understood to be between the molar lattice specific heats of the two solids, with the same number of vibrational degrees of freedom per mole. Their electronic contributions, if any, are neglected.

SOLUTION:

For a three-dimensional Debye solid at $T \ll \Theta_D$, the molar heat capacity at constant volume is

$$C_V = \frac{12\pi^4}{5} R \left(\frac{T}{\Theta_D} \right)^3.$$

At the same temperature, all factors except the Debye temperature cancel in the ratio:

$$\frac{C_{V,A}}{C_{V,B}} = \frac{(T/\Theta_{D,A})^3}{(T/\Theta_{D,B})^3} = \left(\frac{\Theta_{D,B}}{\Theta_{D,A}} \right)^3.$$

Substituting the given Debye temperatures,

$$\frac{C_{V,A}}{C_{V,B}} = \left(\frac{300 \text{ K}}{200 \text{ K}} \right)^3 = \left(\frac{3}{2} \right)^3 = \frac{27}{8} = 3.375.$$

Numerically, using

$$\frac{12\pi^4}{5} R \approx 1.944 \times 10^3 \text{ J mol}^{-1} \text{ K}^{-1},$$

one obtains

$$C_{V,A} \approx 1.944 \times 10^3 \left(\frac{20}{200} \right)^3 = 1.944 \text{ J mol}^{-1} \text{ K}^{-1},$$

and

$$C_{V,B} \approx 1.944 \times 10^3 \left(\frac{20}{300} \right)^3 = 0.576 \text{ J mol}^{-1} \text{ K}^{-1}.$$

Thus the solid with the lower Debye temperature has the larger low-temperature lattice heat capacity, because a larger fraction of its phonon modes is thermally excited.

CONCLUSION:

At 20 K, within the low-temperature Debye approximation,

$$C_{V,A} : C_{V,B} = 27 : 8$$

or

$$C_{V,A} = 3.375 C_{V,B}.$$

The corresponding molar lattice heat capacities are approximately

$$C_{V,A} \approx 1.94 \text{ J mol}^{-1} \text{ K}^{-1}, \quad C_{V,B} \approx 0.576 \text{ J mol}^{-1} \text{ K}^{-1}.$$

Therefore, solid *A* has about 3.4 times the specific heat of solid *B* at the stated temperature, provided lattice vibrations dominate and $T \ll \Theta_D$.

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Q3. Calculate the pressure at which water will boil at 150°C , given that the change in specific volume when 1 gram of water is converted into steam is 1676 cm^3 . Given, latent heat of vaporization for steam = 540 cal per gram , $J = 4.2 \times 10^7 \text{ ergs/cal}$ and one atmospheric pressure = 10^6 dynes/cm^2 .

INTRODUCTION:

The required quantity is the absolute pressure p_2 at which liquid water and steam coexist in equilibrium at

$$T_2 = 150^\circ \text{C} = 423 \text{ K}.$$

At the normal boiling point,

$$T_1 = 100^\circ \text{C} = 373 \text{ K}, \quad p_1 = 1 \text{ atm} = 10^6 \text{ dyn cm}^{-2}.$$

For liquid–vapour equilibrium, the variation of saturation pressure with temperature is governed by the Clapeyron equation

$$\frac{dp}{dT} = \frac{L}{T\Delta v},$$

where L is the latent heat of vaporization per unit mass and

$$\Delta v = v_{\text{steam}} - v_{\text{water}}.$$

The sign convention is

$$\Delta v > 0, \quad \Delta p = p_2 - p_1.$$

Thus $dp/dT > 0$, as expected for vaporization.

Because only one value each of L and Δv is supplied, both are assumed constant over the interval 373–423 K. The phases are assumed to remain in thermodynamic equilibrium along the saturation curve.

SOLUTION:

The latent heat in cgs mechanical units is

$$L = 540 \frac{\text{cal}}{\text{g}} \times 4.2 \times 10^7 \frac{\text{erg}}{\text{cal}} = 2.268 \times 10^{10} \text{ erg g}^{-1}.$$

The given specific-volume change is

$$\Delta v = 1676 \text{ cm}^3 \text{ g}^{-1}.$$

With L and Δv constant, the Clapeyron equation must be integrated while retaining its $1/T$ dependence:

$$dp = \frac{L}{\Delta v} \frac{dT}{T}.$$

Integrating between the normal boiling point (p_1, T_1) and the required state (p_2, T_2),

$$\int_{p_1}^{p_2} dp = \frac{L}{\Delta v} \int_{T_1}^{T_2} \frac{dT}{T},$$

and hence

$$p_2 - p_1 = \frac{L}{\Delta v} \ln\left(\frac{T_2}{T_1}\right).$$

Now,

$$\frac{L}{\Delta v} = \frac{2.268 \times 10^{10} \text{ erg g}^{-1}}{1676 \text{ cm}^3 \text{ g}^{-1}} = 1.3532 \times 10^7 \text{ erg cm}^{-3}.$$

Since

$$1 \text{ erg} = 1 \text{ dyne cm},$$

it follows that

$$1 \text{ erg cm}^{-3} = 1 \text{ dyne cm}^{-2},$$

so

$$\frac{L}{\Delta v} = 1.3532 \times 10^7 \text{ dyn cm}^{-2}.$$

Therefore,

$$p_2 - p_1 = 1.3532 \times 10^7 \ln\left(\frac{423}{373}\right) \text{ dyn cm}^{-2}.$$

Using

$$\ln\left(\frac{423}{373}\right) = \ln(1.13405) \approx 0.12579,$$

we obtain

$$p_2 - p_1 \approx (1.3532 \times 10^7)(0.12579) \text{ dyn cm}^{-2} \approx 1.702 \times 10^6 \text{ dyn cm}^{-2}.$$

Thus

$$p_2 = p_1 + (p_2 - p_1) = (1.000 + 1.702) \times 10^6 \text{ dyn cm}^{-2},$$

or

$$p_2 \approx 2.702 \times 10^6 \text{ dyn cm}^{-2}.$$

Since $1 \text{ atm} = 10^6 \text{ dyn cm}^{-2}$,

$$p_2 \approx 2.70 \text{ atm}.$$

The result has the required physical sign: because $L > 0$, $\Delta v > 0$, and $T_2 > T_1$, the saturation pressure increases. Moreover, for a very small temperature change,

$$\ln\left(\frac{T_2}{T_1}\right) \approx \frac{T_2 - T_1}{T_1},$$

so the integrated result reduces to the local Clapeyron form, as it should.

CONCLUSION:

Under the approximation that the latent heat and specific-volume change remain constant between 100°C and 150°C, water boils at 150°C when the absolute pressure is

$$p_2 \approx 2.70 \times 10^6 \text{ dyn cm}^{-2} \approx 2.70 \text{ atm}.$$

Thus the boiling point rises to 150°C when the external pressure is increased to approximately 2.70 atmospheres.

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Q4. Why does the specific heat of solids depend on material at low temperatures but become independent of material at high temperatures?

INTRODUCTION:

The temperature dependence follows from the quantization of lattice vibrations. A crystalline solid containing N atoms has $3N$ normal vibrational modes, which may be treated, in the harmonic approximation, as independent quantum oscillators of angular

frequencies ω_s . Their frequencies depend on the elastic constants, atomic masses, crystal structure, and interatomic forces; hence the vibrational spectrum is material-dependent.

Let C_V denote the heat capacity at constant volume, k_B the Boltzmann constant, and

$$\Theta_D = \frac{\hbar\omega_D}{k_B}$$

the Debye temperature corresponding to the maximum Debye frequency ω_D . The low- and high-temperature regimes mean $T \ll \Theta_D$ and $T \gg \Theta_D$, respectively. The discussion primarily concerns the lattice heat capacity of a non-magnetic crystalline solid. In metals, an additional electronic contribution is relevant at sufficiently low temperatures.

SOLUTION:

For a normal mode of frequency ω , the allowed energies are

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega, \quad n = 0, 1, 2, \dots$$

Its temperature-dependent mean energy is

$$\bar{E}(\omega, T) = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\hbar\omega/(k_B T)} - 1}.$$

The zero-point term $\hbar\omega/2$ is independent of T and therefore does not contribute to the heat capacity. Differentiation gives the heat capacity of one mode:

$$c_\omega = k_B \left(\frac{\hbar\omega}{k_B T} \right)^2 \frac{e^{\hbar\omega/(k_B T)}}{(e^{\hbar\omega/(k_B T)} - 1)^2}.$$

At low temperature, a mode with $\hbar\omega \gg k_B T$ has an exponentially small thermal occupation. Thus only long-wavelength acoustic modes of sufficiently low frequency are appreciably excited. In the Debye approximation, their density of states is

$$g(\omega) = \frac{9N}{\omega_D^3} \omega^2, \quad 0 \leq \omega \leq \omega_D,$$

where the normalization satisfies

$$\int_0^{\omega_D} g(\omega) d\omega = 3N.$$

The thermal lattice energy is therefore

$$U_{\text{th}} = \int_0^{\omega_D} \frac{\hbar\omega}{e^{\hbar\omega/(k_B T)} - 1} g(\omega) d\omega.$$

On introducing

$$x = \frac{\hbar\omega}{k_B T}, \quad \frac{\hbar\omega_D}{k_B T} = \frac{\Theta_D}{T},$$

one obtains

$$U_{\text{th}} = 9Nk_B T \left(\frac{T}{\Theta_D} \right)^3 \int_0^{\Theta_D/T} \frac{x^3}{e^x - 1} dx.$$

Hence

$$C_V = 9Nk_B \left(\frac{T}{\Theta_D} \right)^3 \int_0^{\Theta_D/T} \frac{x^4 e^x}{(e^x - 1)^2} dx.$$

For $T \ll \Theta_D$, the upper limit can be replaced by infinity. Using

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15},$$

the energy and heat capacity become

$$U_{\text{th}} = \frac{3\pi^4}{5} Nk_B \frac{T^4}{\Theta_D^3},$$

and

$$C_V = \frac{12\pi^4}{5} Nk_B \left(\frac{T}{\Theta_D} \right)^3.$$

Thus the Debye T^3 law is obtained. Since

$$\omega_D = v_D (6\pi^2 n)^{1/3}, \quad \Theta_D = \frac{\hbar v_D}{k_B} (6\pi^2 n)^{1/3},$$

where $n = N/V$ is the number density and v_D is an appropriate average sound velocity, Θ_D depends on density, atomic mass, and elastic constants. Consequently, the coefficient of T^3 ,

$$\frac{C_V}{T^3} = \frac{12\pi^4}{5} \frac{Nk_B}{\Theta_D^3},$$

is different for different materials. This is the origin of the strong material dependence at low temperature.

For a metal, thermally excited electrons near the Fermi surface add a term proportional to T , so that at sufficiently low temperature

$$C_V = \gamma T + \beta T^3, \quad \beta = \frac{12\pi^4}{5} \frac{Nk_B}{\Theta_D^3}.$$

Both γ , which depends on the electronic density of states at the Fermi energy, and β are material-dependent.

At high temperature, $T \gg \Theta_D$, one has

$$x = \frac{\hbar\omega}{k_B T} \ll 1$$

for essentially every lattice mode. Since

$$e^x - 1 \simeq x,$$

the mean thermal energy of each mode approaches

$$\frac{\hbar\omega}{e^{\hbar\omega/(k_B T)} - 1} \simeq k_B T.$$

This is also the classical equipartition result: each harmonic mode has one quadratic kinetic term and one quadratic potential term, contributing

$$\frac{1}{2}k_B T + \frac{1}{2}k_B T = k_B T.$$

Because a solid of N atoms has $3N$ vibrational modes,

$$U_{th} \approx 3Nk_B T,$$

and hence

$$C_V = \left(\frac{\partial U}{\partial T} \right)_V \approx 3Nk_B.$$

For one mole, $N = N_A$, and therefore

$$C_{V,m} \approx 3N_A k_B = 3R \approx 24.9 \text{ J mol}^{-1} \text{ K}^{-1}.$$

This is the Dulong–Petit law. The mode frequencies, and hence the microscopic material properties, disappear from the leading high-temperature result because all $3N$ modes are fully thermally excited and each contributes the same classical amount k_B .

The statement of material independence applies to heat capacity per atom or per mole of atoms. If “specific heat” means heat capacity per unit mass, then

$$c_V = \frac{C_{V,m}}{M} \approx \frac{3R}{M},$$

where M is the molar mass. It therefore retains a dependence on molar mass even in the classical high-temperature limit. Moreover, strong anharmonicity, thermal expansion, phase changes, or differences between C_P and C_V can produce corrections to the ideal Dulong–Petit value.

CONCLUSION:

At low temperatures, only low-frequency acoustic phonons are excited, and their spectrum is fixed by material-specific atomic masses, number density, elastic constants, and sound velocities. Consequently,

$$C_V \approx \frac{12\pi^4}{5} Nk_B \left(\frac{T}{\Theta_D} \right)^3, \quad T \ll \Theta_D,$$

with an additional material-dependent electronic term γT in metals.

At high temperatures, every one of the $3N$ lattice modes reaches the classical equipartition limit and contributes k_B , giving

$$C_V \approx 3Nk_B, \quad C_{V,m} \approx 3R \approx 24.9 \text{ J mol}^{-1} \text{ K}^{-1}, \quad T \gg \Theta_D.$$

Thus the high-temperature lattice heat capacity per atom, or molar heat capacity per mole of atoms, is approximately independent of material because it depends only on the number of vibrational degrees of freedom. The heat capacity per unit mass remains $3R/M$ and is not strictly material-independent.

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Q5. A reversible heat engine operates with three reservoirs at 300 K, 400 K and 1200 K. It absorbs 1200 kJ energy as heat from the reservoir at 1200 K and delivers 400 kJ work. Determine the heat interactions with the other two reservoirs.

INTRODUCTION:

The heat interactions are determined by applying both the first law of thermodynamics and the Clausius equality for a reversible cyclic heat engine.

Let Q_T denote the heat transferred to the engine from a reservoir at absolute temperature T . The sign convention is

$$Q_T > 0$$

for heat absorbed by the engine

$$Q_T < 0$$

for heat rejected by the engine.

Let $W > 0$ denote work delivered by the engine. The given quantities are

$$Q_{1200} = 1200 \text{ kJ}, \quad W = 400 \text{ kJ}.$$

The unknown heat interactions are Q_{300} and Q_{400} . The engine is assumed to complete a thermodynamic cycle, so its changes in internal energy and entropy over one cycle are both zero. Since the engine is reversible, the total entropy change of the reservoirs and engine is zero.

SOLUTION:

For a cyclic heat engine, the first law gives

$$\sum Q = W.$$

Therefore,

$$Q_{300} + Q_{400} + Q_{1200} = W.$$

Substituting the given values,

$$Q_{300} + Q_{400} + 1200 \text{ kJ} = 400 \text{ kJ},$$

and hence

$$Q_{300} + Q_{400} = -800 \text{ kJ}. \quad (1)$$

Because the cycle is reversible, the Clausius equality is

$$\oint \frac{\delta Q_{rev}}{T} = 0.$$

As each heat interaction occurs with a constant-temperature reservoir,

$$\frac{Q_{300}}{300 \text{ K}} + \frac{Q_{400}}{400 \text{ K}} + \frac{Q_{1200}}{1200 \text{ K}} = 0.$$

Substituting $Q_{1200} = 1200 \text{ kJ}$,

$$\frac{Q_{300}}{300 \text{ K}} + \frac{Q_{400}}{400 \text{ K}} + \frac{1200 \text{ kJ}}{1200 \text{ K}} = 0,$$

so that

$$\frac{Q_{300}}{300 \text{ K}} + \frac{Q_{400}}{400 \text{ K}} = -1 \text{ kJ K}^{-1}. \quad (2)$$

Multiplying equation (2) by 1200 K gives

$$4Q_{300} + 3Q_{400} = -1200 \text{ kJ}. \quad (3)$$

Multiplying equation (1) by 3 gives

$$3Q_{300} + 3Q_{400} = -2400 \text{ kJ}. \quad (4)$$

Subtracting equation (4) from equation (3),

$$Q_{300} = (-1200 + 2400) \text{ kJ} = 1200 \text{ kJ}.$$

Using equation (1),

$$Q_{400} = -800 \text{ kJ} - 1200 \text{ kJ} = -2000 \text{ kJ}.$$

The energy balance is verified by

$$1200 \text{ kJ} + 1200 \text{ kJ} - 2000 \text{ kJ} = 400 \text{ kJ} = W.$$

Thus, the total heat absorbed is 2400 kJ, of which 2000 kJ is rejected and the remaining 400 kJ appears as work.

The entropy balance is

$$\frac{1200}{1200} + \frac{1200}{300} - \frac{2000}{400} = (1 + 4 - 5) \text{ kJ K}^{-1} = 0,$$

as required for a reversible cycle. Every term Q/T has units of entropy, kJ K^{-1} , so the entropy equation is dimensionally consistent.

The positive value of Q_{300} means that the engine absorbs heat even from the lowest-temperature reservoir. This is physically possible because the device is not a simple two-reservoir heat engine: part of the high-temperature energy enables reversible heat transfer from the 300 K reservoir to the 400 K reservoir while the composite device delivers net work.

CONCLUSION:

Subject to the stated assumptions of cyclic operation and complete reversibility,

$$Q_{300} = +1200 \text{ kJ}, \quad Q_{400} = -2000 \text{ kJ}.$$

Therefore, the engine absorbs 1200 kJ of heat from the 300 K reservoir and rejects 2000 kJ of heat to the 400 K reservoir. Together with the given absorption of 1200 kJ from the 1200 K reservoir, these interactions conserve energy, produce 400 kJ of work, and generate zero net entropy.

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Q6. A ternary system consists of three components (A , B and C) in equilibrium with two phases. Determine the number of degrees of freedom using the Gibb's phase rule and discuss the effect of pressure and temperature variations on the phase equilibrium.

INTRODUCTION:

The demand is to determine the number of independent intensive variables required to specify the equilibrium state and to explain how variations of temperature and pressure affect the two-phase equilibrium.

Assume that the system is non-reacting, that A , B , and C are chemically independent components, and that the two phases are homogeneous bulk phases in complete thermal, mechanical, and chemical equilibrium. Surface, gravitational, electric, and magnetic effects are neglected.

Let C = number of independent components,

P = number of phases,

F = number of degrees of freedom.

For the present system,

$$C = 3, \quad P = 2.$$

The phases will be denoted by 1 and 2, with mole fractions

$$x_i^{(1)}, \quad x_i^{(2)}, \quad i = A, B, C.$$

Temperature T and pressure p are common to both phases. The thermodynamic pressure convention is that $dp > 0$ denotes an increase in pressure. Differences between the partial molar properties of the phases are defined by

$$\Delta \bar{s}_i = \bar{s}_i^{(2)} - \bar{s}_i^{(1)}, \quad \Delta \bar{v}_i = \bar{v}_i^{(2)} - \bar{v}_i^{(1)}.$$

The governing principle is Gibbs' phase rule,

$$F = C - P + 2,$$

which follows from counting the independent intensive variables and subtracting the independent chemical-equilibrium conditions.

SOLUTION:

For each phase α , the mole fractions satisfy

$$x_A^{(\alpha)} + x_B^{(\alpha)} + x_C^{(\alpha)} = 1.$$

Therefore, only $C - 1$ mole fractions are independent in each phase. Before imposing chemical equilibrium, the complete intensive description contains:

1. the common temperature T ;
2. the common pressure p ;
3. $C - 1$ independent composition variables for each of the P phases.

Thus, the number of intensive variables before imposing the equilibrium constraints is

$$N_{\text{var}} = 2 + P(C - 1).$$

Chemical equilibrium requires that every component have the same chemical potential in all coexisting phases:

$$\mu_i^{(1)} = \mu_i^{(2)} = \dots = \mu_i^{(P)}, \quad i = 1, 2, \dots, C.$$

For any one component, equality among P phases supplies $P - 1$ independent equations. Consequently, the number of independent chemical-equilibrium constraints is

$$N_{\text{con}} = C(P - 1).$$

The number of degrees of freedom is therefore

$$\begin{aligned} F &= N_{\text{var}} - N_{\text{con}} \\ &= [2 + P(C - 1)] - C(P - 1) \\ &= 2 + PC - P - CP + C \\ &= C - P + 2. \end{aligned}$$

For the ternary two-phase system,

$$F = 3 - 2 + 2 = 3.$$

Hence the system is *trivariant*. Three independent intensive variables can be specified while retaining two-phase equilibrium. One convenient choice is T , p , and one independent parameter identifying the pair of equilibrium phase compositions. Once these three variables are fixed, the remaining intensive composition variables are determined by

$$\mu_A^{(1)} = \mu_A^{(2)}, \quad \mu_B^{(1)} = \mu_B^{(2)}, \quad \mu_C^{(1)} = \mu_C^{(2)}.$$

The result under commonly imposed restrictions is as follows:

1. If both temperature and pressure are variable, then

$$F = 3.$$

2. If pressure is fixed externally, one degree of freedom has already been specified. The reduced phase rule gives

$$F_p = C - P + 1 = 3 - 2 + 1 = 2.$$

Thus, temperature and one independent compositional parameter may be varied within the two-phase equilibrium region.

3. If temperature is fixed externally,

$$F_T = C - P + 1 = 2.$$

Pressure and one independent compositional parameter remain variable.

4. If both temperature and pressure are fixed,

$$F_{T,p} = C - P = 3 - 2 = 1.$$

Thus there remains one independent intensive compositional parameter. The compositions of the two phases cannot be chosen independently; they must satisfy all three chemical-potential equalities.

The total or overall composition is distinct from the equilibrium composition of either phase. Once the equilibrium phase compositions are known, changing the overall composition along the appropriate two-phase combination generally changes the relative amounts of the phases rather than their equilibrium intensive compositions. If n_1 and n_2 are the amounts of the two phases, material balance gives

$$z_i = \frac{n_1 x_i^{(1)} + n_2 x_i^{(2)}}{n_1 + n_2}, \quad i = A, B, C,$$

where z_i is the overall mole fraction. Thus, phase amounts are extensive variables and are not counted in F .

For each component in a multicomponent phase,

$$d\mu_i = -\bar{s}_i dT + \bar{v}_i dp + \sum_{j=1}^{C-1} \left(\frac{\partial \mu_i}{\partial x_j} \right)_{T,p} dx_j,$$

where $\bar{s}_i$ and $\bar{v}_i$ are the partial molar entropy and partial molar volume of component i . Subtracting this relation for phase 1 from that for phase 2 gives

$$\begin{aligned} d(\mu_i^{(2)} - \mu_i^{(1)}) &= -\Delta \bar{s}_i dT + \Delta \bar{v}_i dp \\ &+ \sum_{j=1}^{C-1} \left(\frac{\partial \mu_i^{(2)}}{\partial x_j^{(2)}} \right)_{T,p} dx_j^{(2)} - \sum_{j=1}^{C-1} \left(\frac{\partial \mu_i^{(1)}}{\partial x_j^{(1)}} \right)_{T,p} dx_j^{(1)}. \end{aligned}$$

For continued two-phase equilibrium this expression must vanish separately for

$$i = A, \quad B, \quad C,$$

because

$$\mu_i^{(2)} - \mu_i^{(1)} = 0.$$

At fixed pressure,

$$dp = 0,$$

so a change in temperature produces the direct contribution

$$d(\mu_i^{(2)} - \mu_i^{(1)})_{\text{fixed compositions}} = -\Delta \bar{s}_i dT.$$

Hence increasing T lowers the chemical potential of component i more strongly in the phase in which its partial molar entropy is larger. To restore equality of all three chemical potentials, the compositions $x_i^{(1)}$ and $x_i^{(2)}$ generally readjust. Therefore, temperature variation can shift the equilibrium phase compositions, alter the two-phase region, and, for a specified overall composition, change the relative amounts of the phases. If a phase boundary is crossed, one phase can disappear or an additional stable phase can appear.

At fixed temperature,

$$dT = 0,$$

and the direct pressure contribution is

$$d(\mu_i^{(2)} - \mu_i^{(1)})_{\text{fixed compositions}} = \Delta \bar{v}_i dp.$$

Thus increasing pressure raises the chemical potential of component i more strongly in the phase in which its partial molar volume is larger. The equilibrium compositions must again change so that

$$\mu_i^{(1)} = \mu_i^{(2)}$$

remains satisfied for every component. Pressure effects are usually relatively small when both phases are condensed and have similar partial molar volumes, but they may be substantial when one phase is gaseous or when the phases have appreciably different densities.

No universal statement that an entire phase is favoured merely because it has a larger molar entropy or molar volume can be made for arbitrary ternary phases of different compositions. The correct coexistence criterion is the component-wise equality of chemical potentials. A definite prediction of the direction and magnitude of the composition shifts requires the partial molar entropies, partial molar volumes, and composition dependence of the three chemical potentials.

As a dimensional check,

$$[\mu_i] = \text{J mol}^{-1},$$

while

$$[\bar{s}_i dT] = \text{J mol}^{-1} \text{K}^{-1} \text{K} = \text{J mol}^{-1},$$

and

$$[\bar{v}_i dp] = \text{m}^3 \text{mol}^{-1} \text{Pa} = \text{J mol}^{-1}.$$

Thus every term in $d\mu_i$ has the correct dimensions.

As a limiting check, setting $C = 1$ and $P = 2$ gives

$$F = 1 - 2 + 2 = 1,$$

which is the familiar result that two phases of a pure substance coexist along a one-parameter curve: choosing either T or p fixes the other. This confirms the consistency of the phase-rule counting.

CONCLUSION:

For a non-reacting ternary system containing two homogeneous phases in equilibrium,

$$F = C - P + 2 = 3.$$

The system is therefore trivariant. The corresponding restricted degrees of freedom are

$$F_p = 2, \quad F_T = 2, \quad F_{T,p} = 1.$$

Temperature affects equilibrium through the partial molar entropy terms

$$\left(\frac{\partial \mu_i}{\partial T}\right)_{p,x} = -\bar{s}_i,$$

whereas pressure affects it through the partial molar volume terms

$$\left(\frac{\partial \mu_i}{\partial p}\right)_{T,x} = \bar{v}_i.$$

Consequently, variations in T or p generally shift the compositions of the two coexisting phases so that

$$\mu_i^{(1)} = \mu_i^{(2)}, \quad i = A, B, C$$

continues to hold. For a specified overall composition, these shifts can also alter the relative phase amounts, and sufficiently large variations can cause a phase boundary to be crossed. The result is valid provided there are three independent non-reacting components, two bulk equilibrium phases, and no additional externally imposed thermodynamic constraints.

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Q7. Discuss briefly the considerations which led Van der Waals to modify the gas equation. What are the critical constants of a gas? Calculate the values of these constants in terms of the constants of the Van der Waals equation.

INTRODUCTION:

The ideal-gas equation for one mole,

$$PV_m = RT,$$

is based on two simplifying assumptions: the molecules occupy negligible volume, and no intermolecular forces act between them except during perfectly elastic collisions. Real gases depart appreciably

from this equation at high pressure and low temperature, where molecular size and intermolecular attraction become important.

Van der Waals therefore introduced two corrections. If b is the excluded molar volume, the volume actually available for molecular motion is $V_m - b$, rather than V_m . If a measures the strength of intermolecular attraction, the observed pressure is smaller than the pressure that would exist without attraction. The corresponding pressure correction for one mole is a/V_m^2 . Thus the Van der Waals equation is

$$\left(P + \frac{a}{V_m^2}\right)(V_m - b) = RT,$$

or

$$P = \frac{RT}{V_m - b} - \frac{a}{V_m^2}.$$

Here P is the pressure, T the absolute temperature, V_m the molar volume, R the universal gas constant, and $a, b > 0$ are characteristic constants of the gas.

The critical constants are the critical temperature T_c , critical pressure P_c , and critical molar volume V_c . At T_c , the distinction between the liquid and vapour phases disappears. The critical isotherm has a stationary point of inflection at (P_c, V_c) ; hence

$$\left(\frac{\partial P}{\partial V_m}\right)_{T_c, V_c} = 0, \quad \left(\frac{\partial^2 P}{\partial V_m^2}\right)_{T_c, V_c} = 0.$$

SOLUTION:

The reasons for modifying the ideal-gas equation are as follows.

1. Finite molecular volume:

Real molecules are not point particles. At sufficiently high pressure, their own volume is no longer negligible compared with the volume of the vessel. Moreover, because two molecules cannot occupy the same region of space, the volume available for translational motion is smaller than the measured molar volume. Van der Waals represented this reduction by replacing

$$V_m \longrightarrow V_m - b.$$

2. Intermolecular attraction:

A molecule well inside the gas is attracted approximately equally in all directions, so the resultant force on it is nearly zero. A molecule approaching a wall, however, is attracted predominantly back into the gas. Its momentum transfer to the wall, and hence the measured pressure, is therefore reduced.

The reduction is proportional both to the number density of molecules near the wall and

to the number density of molecules attracting them. It consequently varies as the square of the density:

$$\Delta P \propto \left(\frac{1}{V_m}\right)^2.$$

Writing the proportionality constant as a ,

$$P_{\text{ideal interaction-free}} = P + \frac{a}{V_m^2}.$$

Combining the pressure and volume corrections gives

$$\left(P + \frac{a}{V_m^2}\right)(V_m - b) = RT.$$

To calculate the critical constants, write the equation of state explicitly as

$$P(V_m, T) = \frac{RT}{V_m - b} - \frac{a}{V_m^2}.$$

At the critical point, the first derivative of pressure with respect to molar volume at constant temperature vanishes:

$$\left(\frac{\partial P}{\partial V_m}\right)_T = -\frac{RT}{(V_m - b)^2} + \frac{2a}{V_m^3}.$$

Therefore,

$$-\frac{RT_c}{(V_c - b)^2} + \frac{2a}{V_c^3} = 0,$$

so that

$$\frac{RT_c}{(V_c - b)^2} = \frac{2a}{V_c^3}. \quad (1)$$

The second derivative is

$$\left(\frac{\partial^2 P}{\partial V_m^2}\right)_T = \frac{2RT}{(V_m - b)^3} - \frac{6a}{V_m^4}.$$

At the critical point,

$$\frac{2RT_c}{(V_c - b)^3} - \frac{6a}{V_c^4} = 0,$$

or

$$\frac{RT_c}{(V_c - b)^3} = \frac{3a}{V_c^4}. \quad (2)$$

Dividing equation (2) by equation (1) eliminates T_c :

$$\frac{1}{V_c - b} = \frac{3a/V_c^4}{2a/V_c^3} = \frac{3}{2V_c}.$$

Hence

$$2V_c = 3(V_c - b),$$

which gives

$$\boxed{V_c = 3b}.$$

Substituting $V_c = 3b$ into equation (1),

$$\frac{RT_c}{(3b - b)^2} = \frac{2a}{(3b)^3}.$$

Thus

$$\frac{RT_c}{4b^2} = \frac{2a}{27b^3},$$

and therefore

$$T_c = \frac{8a}{27Rb}.$$

The critical pressure follows from the equation of state:

$$P_c = \frac{RT_c}{V_c - b} - \frac{a}{V_c^2}.$$

Using $V_c = 3b$ and $RT_c = 8a/(27b)$,

$$P_c = \frac{8a/(27b)}{3b - b} - \frac{a}{(3b)^2}.$$

Therefore,

$$P_c = \frac{8a}{54b^2} - \frac{a}{9b^2} = \frac{4a}{27b^2} - \frac{3a}{27b^2},$$

so

$$P_c = \frac{a}{27b^2}.$$

These expressions satisfy the critical-point derivative conditions explicitly. For example,

$$\left(\frac{\partial P}{\partial V_m} \right)_c = -\frac{RT_c}{(V_c - b)^2} + \frac{2a}{V_c^3} = -\frac{2a}{27b^3} + \frac{2a}{27b^3} = 0,$$

and

$$\left(\frac{\partial^2 P}{\partial V_m^2} \right)_c = \frac{2RT_c}{(V_c - b)^3} - \frac{6a}{V_c^4} = \frac{2a}{27b^4} - \frac{2a}{27b^4} = 0.$$

The dimensional checks are

$$[a] = [P][V_m]^2, \quad [b] = [V_m].$$

Consequently,

$$\left[\frac{a}{b^2} \right] = [P], \quad \left[\frac{a}{Rb} \right] = [T],$$

as required for P_c and T_c .

A further consistency check is the critical compressibility factor:

$$Z_c = \frac{P_c V_c}{RT_c} = \frac{(a/27b^2)(3b)}{8a/(27b)} = \frac{3}{8}.$$

Thus the Van der Waals equation predicts the same dimensionless critical compressibility factor for all gases. In the dilute-gas limit $V_m \gg b$, with a/V_m^2 negligible, the equation reduces to

$$PV_m \simeq RT,$$

recovering ideal-gas behaviour.

CONCLUSION:

Van der Waals modified the ideal-gas equation to account for the finite size of molecules through the excluded-volume constant b , and for intermolecular attraction through the pressure correction a/V_m^2 . For one mole,

$$\left(P + \frac{a}{V_m^2} \right) (V_m - b) = RT.$$

The critical constants obtained from the stationary-inflection conditions on the critical isotherm are

$$V_c = 3b, \quad P_c = \frac{a}{27b^2}, \quad T_c = \frac{8a}{27Rb}.$$

They also obey

$$\frac{P_c V_c}{RT_c} = \frac{3}{8}.$$

Hence the Van der Waals parameters determine the pressure, temperature, and molar volume at which the liquid and gaseous phases become indistinguishable.

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Q8. Explain why, at equilibrium, the chemical potential of a component must be the same in all coexisting phases. Derive the equilibrium condition for a binary liquid-vapour system in terms of chemical potential.

INTRODUCTION:

The chemical potential of component i in a phase is the partial molar Gibbs free energy,

$$\mu_i = \left(\frac{\partial G}{\partial n_i} \right)_{T, P, n_{j \neq i}}.$$

It therefore measures the change in Gibbs free energy when an infinitesimal amount of that component is added to the phase at fixed temperature and pressure.

Consider a closed, non-reacting system containing homogeneous coexisting phases. The phases are assumed to have a common temperature T and pressure P , and interfacial, gravitational, and other finite-size contributions are neglected. The total amount of each component is conserved, but matter may pass between phases. At fixed T and P , stable equilibrium requires the total Gibbs free energy to be stationary against every allowed infinitesimal transfer and to be

a minimum with respect to finite admissible variations.

For the binary liquid-vapour system, let the components be 1 and 2, and use superscripts l and v for the liquid and vapour phases. Thus,

$$\mu_i^l = \mu_i^l(T, P, x_1, x_2), \quad \mu_i^v = \mu_i^v(T, P, y_1, y_2),$$

where x_i and y_i are the liquid- and vapour-phase mole fractions, with

$$x_1 + x_2 = 1, \quad y_1 + y_2 = 1.$$

The required equilibrium conditions follow by minimizing the total Gibbs free energy subject to conservation of both components.

SOLUTION:

For a multicomponent homogeneous phase, the fundamental Gibbs relation is

$$dG = -S dT + V dP + \sum_i \mu_i dn_i.$$

At fixed T and P , this reduces to

$$dG = \sum_i \mu_i dn_i.$$

Consider first one component i distributed between two coexisting phases α and β . Let a positive infinitesimal amount dn_i be transferred from phase α to phase β . Conservation of the total amount of component i gives

$$dn_i^\alpha = -dn_i, \quad dn_i^\beta = +dn_i.$$

The corresponding change in the total Gibbs free energy is

$$\begin{aligned} dG_{\text{tot}} &= \mu_i^\alpha dn_i^\alpha + \mu_i^\beta dn_i^\beta \\ &= -\mu_i^\alpha dn_i + \mu_i^\beta dn_i \\ &= (\mu_i^\beta - \mu_i^\alpha) dn_i. \end{aligned}$$

At equilibrium, an allowed infinitesimal transfer in either direction must produce no first-order change in G_{tot} . Since dn_i is arbitrary,

$$dG_{\text{tot}} = 0 \implies \boxed{\mu_i^\alpha = \mu_i^\beta}.$$

The physical meaning can also be seen from the sign of dG_{tot} . If

$$\mu_i^\alpha > \mu_i^\beta,$$

then transfer from α to β gives

$$dG_{\text{tot}} = (\mu_i^\beta - \mu_i^\alpha) dn_i < 0,$$

so the transfer is spontaneous. Thus a component moves, in the thermodynamic sense, from a phase of

higher chemical potential to one of lower chemical potential. Such transfer ceases only when its chemical potentials are equal. For more than two coexisting phases,

$$\boxed{\mu_i^{(1)} = \mu_i^{(2)} = \dots = \mu_i^{(r)}}$$

for every component i .

Now consider a binary liquid-vapour system. Its total Gibbs free energy is

$$G = G^l(T, P, n_1^l, n_2^l) + G^v(T, P, n_1^v, n_2^v).$$

Because the system is closed and non-reacting, the total amount of each component is fixed:

$$n_1^l + n_1^v = N_1, \quad n_2^l + n_2^v = N_2.$$

Hence the allowed variations satisfy

$$dn_1^v = -dn_1^l, \quad dn_2^v = -dn_2^l.$$

At fixed T and P , the variation of the total Gibbs free energy is

$$dG = \mu_1^l dn_1^l + \mu_2^l dn_2^l + \mu_1^v dn_1^v + \mu_2^v dn_2^v.$$

Using the conservation constraints,

$$\begin{aligned} dG &= \mu_1^l dn_1^l + \mu_2^l dn_2^l - \mu_1^v dn_1^l - \mu_2^v dn_2^l \\ &= (\mu_1^l - \mu_1^v) dn_1^l + (\mu_2^l - \mu_2^v) dn_2^l. \end{aligned}$$

The amounts of components 1 and 2 may be independently redistributed between the two phases while their respective total amounts remain fixed. Therefore dn_1^l and dn_2^l are independent admissible variations. The equilibrium condition $dG = 0$ for every such variation requires the coefficient of each variation to vanish:

$$\boxed{\mu_1^l(T, P, x_1, x_2) = \mu_1^v(T, P, y_1, y_2)},$$

$$\boxed{\mu_2^l(T, P, x_1, x_2) = \mu_2^v(T, P, y_1, y_2)}.$$

Together with

$$T^l = T^v, \quad P^l = P^v,$$

these are the conditions for thermal, mechanical, and diffusive equilibrium, respectively.

The result may equivalently be obtained with constrained minimization. Introduce Lagrange multipliers λ_1 and λ_2 for conservation of the two components:

$$\Phi = G^l + G^v - \lambda_1(n_1^l + n_1^v - N_1) - \lambda_2(n_2^l + n_2^v - N_2).$$

Stationarity with respect to the four independently varied phase amounts gives

$$\frac{\partial \Phi}{\partial n_1^l} = \mu_1^l - \lambda_1 = 0, \quad \frac{\partial \Phi}{\partial n_1^v} = \mu_1^v - \lambda_1 = 0,$$

and

$$\frac{\partial \Phi}{\partial n_2^l} = \mu_2^l - \lambda_2 = 0, \quad \frac{\partial \Phi}{\partial n_2^v} = \mu_2^v - \lambda_2 = 0.$$

Consequently,

$$\mu_1^l = \lambda_1 = \mu_1^v, \quad \mu_2^l = \lambda_2 = \mu_2^v,$$

which reproduces the equilibrium conditions.

For a general non-ideal mixture, the chemical potential may be expressed in terms of fugacity:

$$\mu_i(T, P, \text{composition}) = \mu_i^\circ(T) + RT \ln\left(\frac{f_i}{f^\circ}\right).$$

At the same temperature, equality of chemical potentials is therefore equivalent to equality of component fugacities:

$$\mu_i^l = \mu_i^v \iff f_i^l = f_i^v, \quad i = 1, 2.$$

Thus the binary phase-equilibrium relations can also be written as

$$f_1^l(T, P, x_1, x_2) = f_1^v(T, P, y_1, y_2)$$

$$f_2^l(T, P, x_1, x_2) = f_2^v(T, P, y_1, y_2).$$

As a check, if $\mu_i^l - \mu_i^v > 0$, transferring component i from liquid to vapour gives

$$dG = (\mu_i^v - \mu_i^l) dn_i^v < 0,$$

so vaporization is favoured. If $\mu_i^l - \mu_i^v < 0$, condensation is favoured. When the two chemical potentials are equal, neither direction lowers the Gibbs free energy, as required at phase equilibrium. Dimensionally, μ_i has units of energy per mole, and $\mu_i dn_i$ has units of energy, consistent with its contribution to dG .

CONCLUSION:

At fixed common temperature and pressure, the equilibrium state minimizes the total Gibbs free energy subject to conservation of every component. An infinitesimal transfer of component i between two phases changes the Gibbs free energy by the chemical-potential difference multiplied by the transferred amount. Therefore, diffusive equilibrium requires

$$\mu_i^\alpha = \mu_i^\beta$$

for each component in every pair of coexisting phases.

For a closed, non-reacting binary liquid-vapour system with homogeneous bulk phases and negligible interfacial and gravitational effects, the complete phase-equilibrium conditions are

$$T^l = T^v, \quad P^l = P^v,$$

$$\mu_1^l(T, P, \mathbf{x}) = \mu_1^v(T, P, \mathbf{y}), \quad \mu_2^l(T, P, \mathbf{x}) = \mu_2^v(T, P, \mathbf{y}),$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, $x_1 + x_2 = 1$, and $y_1 + y_2 = 1$. Equivalently,

$$f_1^l = f_1^v, \quad f_2^l = f_2^v.$$

A chemical-potential inequality produces net transfer from the phase of higher chemical potential to that of lower chemical potential; equality signifies that neither vaporization nor condensation of either component can lower the Gibbs free energy.

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Q9. For silver, the specific heat at constant pressure in the range of 50 K to 100 K is given by : $C_p = 0.076 T - 0.00026 T^2 - 0.15 \text{ cal mol}^{-1} \text{ deg}^{-1}$. If 2 moles of silver are heated from 50 K to 100 K, calculate the change in entropy.

INTRODUCTION:

The demand is to calculate the entropy change of $n = 2$ moles of silver when its temperature increases from

$$T_1 = 50 \text{ K} \quad \text{to} \quad T_2 = 100 \text{ K}$$

at constant pressure.

Entropy is a state function. Hence its change may be evaluated along a reversible constant-pressure path between the same equilibrium states. For reversible heating,

$$dS = \frac{\delta Q_{\text{rev}}}{T},$$

and, at constant pressure with no phase transition or additional non- pV work,

$$\delta Q_{\text{rev}} = nC_p(T) dT.$$

Therefore,

$$\Delta S = n \int_{T_1}^{T_2} \frac{C_p(T)}{T} dT.$$

Here $C_p(T)$ is the molar heat capacity, T is the absolute temperature, and $\Delta S = S_2 - S_1$. We use the laboratory thermodynamic frame and the sign convention that heat absorbed by the silver is positive; consequently, heating with $C_p > 0$ should give $\Delta S > 0$. The empirical expression is assumed valid throughout $50 \leq T \leq 100 \text{ K}$, with no phase change in this interval. Since a temperature interval of one kelvin equals one Celsius degree, $\text{cal mol}^{-1} \text{ deg}^{-1}$ may be used as $\text{cal mol}^{-1} \text{ K}^{-1}$.

SOLUTION:

The given molar heat capacity is

$$C_p(T) = 0.076T - 0.00026T^2 - 0.15 \text{ cal mol}^{-1} \text{ K}^{-1}.$$

Dividing by the absolute temperature,

$$\frac{C_p(T)}{T} = 0.076 - 0.00026T - \frac{0.15}{T}.$$

Thus,

$$\Delta S = 2 \int_{50}^{100} \left(0.076 - 0.00026T - \frac{0.15}{T} \right) dT.$$

Integrating term by term gives

$$\Delta S = 2 \left[0.076T - \frac{0.00026}{2}T^2 - 0.15 \ln T \right]_{50}^{100}.$$

Equivalently,

$$\Delta S = 2 \left[0.076(T_2 - T_1) - 0.00013(T_2^2 - T_1^2) - 0.15 \ln \left(\frac{T_2}{T_1} \right) \right].$$

Substituting $T_1 = 50 \text{ K}$ and $T_2 = 100 \text{ K}$,

$$\begin{aligned} \Delta S &= 2 \left[0.076(100 - 50) - 0.00013(100^2 - 50^2) - 0.15 \ln \left(\frac{100}{50} \right) \right] \text{ cal K}^{-1} \\ &= 2 [0.076(50) - 0.00013(10000 - 2500) - 0.15 \ln 2] \text{ cal K}^{-1} \\ &= 2 [3.800 - 0.00013(7500) - 0.15(0.693147)] \text{ cal K}^{-1} \\ &= 2 [3.800 - 0.975 - 0.103972] \text{ cal K}^{-1} \\ &= 2(2.72103) \text{ cal K}^{-1} \\ &= 5.44206 \text{ cal K}^{-1}. \end{aligned}$$

In SI units, using $1 \text{ cal} = 4.184 \text{ J}$,

$$\Delta S = 5.44206 \text{ cal K}^{-1} \times 4.184 \text{ J cal}^{-1} = 22.77 \text{ J K}^{-1}.$$

The dimensions are consistent because

$$n \frac{C_p}{T} dT \sim \text{mol} \frac{\text{cal mol}^{-1} \text{ K}^{-1}}{\text{K}} \text{ K} = \text{cal K}^{-1},$$

which is the correct dimension of entropy.

As a plausibility check,

$$C_p(50) = 0.076(50) - 0.00026(50)^2 - 0.15 = 3.00 \text{ cal mol}^{-1} \text{ K}^{-1},$$

and

$$C_p(100) = 0.076(100) - 0.00026(100)^2 - 0.15 = 4.85 \text{ cal mol}^{-1} \text{ K}^{-1}.$$

Thus $C_p > 0$ throughout the interval, so heating must produce a positive entropy change, in agreement with the calculated sign. Also, in the limiting case $T_2 \rightarrow T_1$, every term in the integrated expression vanishes and $\Delta S \rightarrow 0$, as required.

CONCLUSION:

For 2 moles of silver heated at constant pressure from 50 K to 100 K,

$$\Delta S = +5.44 \text{ cal K}^{-1}$$

or, equivalently,

$$\Delta S = +22.8 \text{ J K}^{-1}.$$

The positive sign means that the entropy of the silver increases during heating. This result is valid provided the stated heat-capacity formula applies over 50–100 K, the initial and final states are equilibrium states, and no phase transition occurs in the interval.

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Q10. One gram of water (1 cm^3) becomes 1671 cm^3 of steam when boiled at constant pressure of 1 atm ($1.013 \times 10^5 \text{ Pa}$). The heat of vapourisation at this pressure is $L = 2.256 \times 10^6 \text{ J/kg}$. Calculate : (i) The work done by the water when it vapourizes, and; (ii) Increase in its internal energy.

INTRODUCTION:

We consider a closed system consisting of $m = 1 \text{ g} = 1.000 \times 10^{-3} \text{ kg}$ of water in the laboratory frame. State 1 is liquid water of volume $V_1 = 1 \text{ cm}^3$, and state 2 is steam of volume $V_2 = 1671 \text{ cm}^3$. The vapourization occurs at the constant external pressure

$$P = 1.013 \times 10^5 \text{ Pa}.$$

The heat supplied to the system is positive, and the boundary work W is defined as positive when done by the water on its surroundings. With this convention, the first law of thermodynamics is

$$\Delta U = Q - W.$$

We neglect changes in the macroscopic kinetic and gravitational potential energies. Since the phase change occurs at constant pressure and the stated latent heat L is for that pressure,

$$Q = mL.$$

The expansion work is obtained from

$$W = \int_{V_1}^{V_2} P_{\text{ext}} dV = P(V_2 - V_1),$$

because the external pressure is constant.

SOLUTION:

(i) The increase in volume is

$$\Delta V = V_2 - V_1 = (1671 - 1) \text{ cm}^3 = 1670 \text{ cm}^3.$$

Using

$$1 \text{ cm}^3 = 10^{-6} \text{ m}^3,$$

we obtain

$$\Delta V = 1670 \times 10^{-6} \text{ m}^3 = 1.670 \times 10^{-3} \text{ m}^3.$$

Therefore, the work done by the water is

$$\begin{aligned} W &= P\Delta V \\ &= (1.013 \times 10^5 \text{ Pa}) (1.670 \times 10^{-3} \text{ m}^3) \\ &= 1.69171 \times 10^2 \text{ Pa m}^3. \end{aligned}$$

Since $1 \text{ Pa m}^3 = 1 \text{ J}$,

$$W = 1.692 \times 10^2 \text{ J} \approx 169.2 \text{ J}.$$

The positive sign is consistent with the physical process: the water expands and does work on the surroundings.

(ii) The heat absorbed during vapourization is

$$\begin{aligned} Q &= mL \\ &= (1.000 \times 10^{-3} \text{ kg}) (2.256 \times 10^6 \text{ J kg}^{-1}) \\ &= 2.256 \times 10^3 \text{ J} \\ &= 2256 \text{ J}. \end{aligned}$$

Applying the first law,

$$\begin{aligned} \Delta U &= Q - W \\ &= 2256 \text{ J} - 169.171 \text{ J} \\ &= 2086.829 \text{ J}. \end{aligned}$$

Hence,

$$\Delta U = 2.087 \times 10^3 \text{ J} \approx 2.087 \text{ kJ}.$$

Equivalently, at constant pressure the latent heat supplies the enthalpy increase:

$$mL = \Delta H = \Delta U + P\Delta V,$$

which reproduces the same result.

The dimensions are consistent:

$$[P\Delta V] = \text{Pa m}^3 = \frac{\text{N}}{\text{m}^2} \text{m}^3 = \text{N m} = \text{J}.$$

Moreover,

$$0 < W < Q,$$

so the internal energy increases while part of the supplied heat is transferred outward as expansion work. Numerically,

$$\frac{W}{Q} \approx 0.0750,$$

so about 7.50% of the latent heat becomes expansion work and about 92.5% increases the internal energy. As a limiting check, if the volume change were negligible, $P\Delta V \rightarrow 0$, the first law would give $\Delta U \rightarrow Q$, as expected.

CONCLUSION:

At $1.013 \times 10^5 \text{ Pa}$, the vapourizing water expands by $1.670 \times 10^{-3} \text{ m}^3$. Thus,

$$(i) \text{ Work done by the water on the surroundings} = 1.692 \times 10^2 \text{ J}$$

and

$$(ii) \text{ Increase in internal energy} = 2.087 \times 10^3 \text{ J} = 2.087 \text{ kJ}.$$

These results assume constant external pressure, negligible changes in bulk kinetic and potential energies, and the stated latent heat at the specified pressure. The water transfers approximately 169.2 J outward as mechanical work, while its internal energy increases by approximately 2.087 kJ; hence most of the absorbed latent heat increases the molecular internal energy of the steam.



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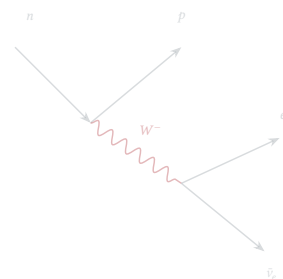
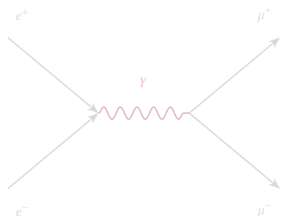
$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$E_n = -\frac{13.6}{n^2} \text{ eV}$$

PREVIOUS YEAR QUESTIONS



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CSE + IFoS

SUBJECT

Quantum Mechanics

10 SOLVED QUESTIONS • CSE + IFoS

$$n \rightarrow p + e^- + \bar{\nu}_e$$

$$E^2 = p^2 c^2 + m^2 c^4$$

$$\partial_\mu j^\mu = 0$$

SUBJECT-WISE • TOPIC-WISE • SOURCE-TAGGED

For Civil Services and Indian Forest Service Examinations

Sample Preview extbar 10 Solved Questions

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CSE • 2026 • 5 MARKS

Q1. The de Broglie wavelength of a particle in thermal equilibrium at a temperature of 27°C is λ . At what temperature will it be $\frac{\lambda}{2}$?

INTRODUCTION:

We have to determine the temperature at which the thermal de Broglie wavelength of the same non-relativistic particle becomes half its initial value. For a particle of mass m ,

$$\lambda_{\text{dB}} = \frac{h}{p},$$

where h is Planck's constant and p is the magnitude of its momentum. In thermal equilibrium, any fixed conventional measure of the characteristic thermal momentum, such as the root-mean-square momentum, is proportional to $\sqrt{T}$, where T is the absolute temperature. Hence the thermal de Broglie wavelength varies as $T^{-1/2}$.

Let

$$T_1 = 27^\circ\text{C} = 300\text{ K}, \quad \lambda_1 = \lambda,$$

and let T_2 be the temperature at which

$$\lambda_2 = \frac{\lambda}{2}.$$

SOLUTION:

For a non-relativistic particle in thermal equilibrium, the equipartition theorem gives

$$\frac{p_{\text{rms}}^2}{2m} = \frac{3}{2}k_{\text{B}}T,$$

so that

$$p_{\text{rms}} = \sqrt{3mk_{\text{B}}T}.$$

Therefore,

$$\lambda_{\text{dB}} = \frac{h}{p_{\text{rms}}} = \frac{h}{\sqrt{3mk_{\text{B}}T}} \propto \frac{1}{\sqrt{T}}.$$

For the same particle at the two temperatures,

$$\frac{\lambda_2}{\lambda_1} = \sqrt{\frac{T_1}{T_2}}.$$

Substituting $\lambda_2 = \lambda/2$, $\lambda_1 = \lambda$, and $T_1 = 300\text{ K}$,

$$\frac{1}{2} = \sqrt{\frac{300}{T_2}}.$$

Squaring both sides,

$$\frac{1}{4} = \frac{300}{T_2},$$

and hence

$$T_2 = 4(300\text{ K}) = 1200\text{ K}.$$

Expressed on the Celsius scale,

$$t_2 = T_2 - 273 \approx 1200 - 273 = 927^\circ\text{C}.$$

Thus halving the wavelength requires the characteristic momentum to double; since $p_{\text{rms}} \propto \sqrt{T}$, the absolute temperature must increase by a factor of four.

CONCLUSION:

The required temperature is

$$T_2 = 1200\text{ K} \approx 927^\circ\text{C}.$$

This result assumes the same particle remains in the non-relativistic thermal regime.

CSE • 2026 • 10 MARKS

Q2. If the position and momentum operators in 1-D are represented by $\hat{x}$ and $\hat{p}$ respectively, calculate the commutator $[\hat{x}^2, \cos(\hat{p})]$.

INTRODUCTION:

The required commutator is evaluated using the canonical position-momentum relation

$$[\hat{x}, \hat{p}] = i\hbar.$$

For any sufficiently differentiable function $f(\hat{p})$, the corresponding operator identity is

$$[\hat{x}, f(\hat{p})] = i\hbar f'(\hat{p}).$$

Here $\cos(\hat{p})$ is understood as an operator function defined by its power series or, equivalently, by the spectral calculus. Its argument is assumed dimensionless; if a momentum scale p_0 is required, $\hat{p}$ is to be replaced by $\hat{p}/p_0$.

SOLUTION:

Using the product rule for commutators,

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B},$$

we obtain

$$[\hat{x}^2, \cos(\hat{p})] = \hat{x}[\hat{x}, \cos(\hat{p})] + [\hat{x}, \cos(\hat{p})]\hat{x}.$$

Since

$$[\hat{x}, \cos(\hat{p})] = i\hbar \frac{d}{d\hat{p}} \cos(\hat{p}) = -i\hbar \sin(\hat{p}),$$

it follows that

$$[\hat{x}^2, \cos(\hat{p})] = -i\hbar [\hat{x} \sin(\hat{p}) + \sin(\hat{p})\hat{x}].$$

Thus the most symmetric form of the answer is

$$[\hat{x}^2, \cos(\hat{p})] = -i\hbar\{\hat{x}, \sin(\hat{p})\},$$

where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ denotes the anticommutator.

An equivalent normally ordered form can be found from

$$[\hat{x}, \sin(\hat{p})] = i\hbar \cos(\hat{p}),$$

which gives

$$\hat{x} \sin(\hat{p}) = \sin(\hat{p})\hat{x} + i\hbar \cos(\hat{p}).$$

Therefore,

$$[\hat{x}^2, \cos(\hat{p})] = -2i\hbar \sin(\hat{p})\hat{x} + \hbar^2 \cos(\hat{p}).$$

Equivalently, with $\hat{x}$ placed on the left,

$$[\hat{x}^2, \cos(\hat{p})] = -2i\hbar \hat{x} \sin(\hat{p}) - \hbar^2 \cos(\hat{p}).$$

CONCLUSION:

The required commutator is

$$[\hat{x}^2, \cos(\hat{p})] = -i\hbar [\hat{x} \sin(\hat{p}) + \sin(\hat{p})\hat{x}] = -i\hbar\{\hat{x}, \sin(\hat{p})\}.$$

The equivalent ordered expressions are

$$[\hat{x}^2, \cos(\hat{p})] = -2i\hbar \sin(\hat{p})\hat{x} + \hbar^2 \cos(\hat{p})$$

and

$$[\hat{x}^2, \cos(\hat{p})] = -2i\hbar \hat{x} \sin(\hat{p}) - \hbar^2 \cos(\hat{p}).$$

CSE • 2026 • 10 MARKS

Q3. A particle is in the normalized state ψ which is a superposition of the energy eigenstates ψ_1 and ψ_2 with energies 10 eV and 30 eV, respectively. The average value of the energy of the particle in the state ψ is 24 eV. Find the state ψ in terms of ψ_1 and ψ_2 .

INTRODUCTION:

Let the Hamiltonian satisfy

$$\hat{H}\psi_1 = E_1\psi_1, \quad \hat{H}\psi_2 = E_2\psi_2,$$

where

$$E_1 = 10 \text{ eV}, \quad E_2 = 30 \text{ eV}.$$

Since eigenstates belonging to distinct eigenvalues of a Hermitian Hamiltonian are orthogonal, we take

$$\langle \psi_i | \psi_j \rangle = \delta_{ij}.$$

The most general superposition is

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle,$$

where $c_1, c_2 \in \mathbb{C}$. Normalization and the given expectation value of energy will determine the probabilities $|c_1|^2$ and $|c_2|^2$; the relative phase cannot be determined from an energy expectation value alone.

SOLUTION:

Normalization gives

$$\langle \psi | \psi \rangle = |c_1|^2 + |c_2|^2 = 1.$$

The expectation value of the Hamiltonian is

$$\begin{aligned} \langle E \rangle &= \langle \psi | \hat{H} | \psi \rangle \\ &= |c_1|^2 E_1 + |c_2|^2 E_2, \end{aligned}$$

because the cross terms vanish by orthogonality. Hence

$$10|c_1|^2 + 30|c_2|^2 = 24,$$

with all energies measured in eV.

Using $|c_1|^2 = 1 - |c_2|^2$,

$$10(1 - |c_2|^2) + 30|c_2|^2 = 24,$$

so that

$$10 + 20|c_2|^2 = 24$$

and therefore

$$|c_2|^2 = \frac{14}{20} = \frac{7}{10}.$$

Consequently,

$$|c_1|^2 = 1 - \frac{7}{10} = \frac{3}{10}.$$

Thus the coefficient magnitudes are

$$|c_1| = \sqrt{\frac{3}{10}}, \quad |c_2| = \sqrt{\frac{7}{10}}.$$

After removing an physically irrelevant common phase, the remaining relative phase may be denoted by ϕ , giving

$$|\psi\rangle = \sqrt{\frac{3}{10}} |\psi_1\rangle + e^{i\phi} \sqrt{\frac{7}{10}} |\psi_2\rangle, \quad \phi \in \mathbb{R}.$$

Indeed, this state yields

$$\langle E \rangle = \frac{3}{10}(10 \text{ eV}) + \frac{7}{10}(30 \text{ eV}) = 3 \text{ eV} + 21 \text{ eV} = 24 \text{ eV},$$

as required.

CONCLUSION:

Up to an arbitrary overall phase, the normalized state is

$$\psi = \sqrt{\frac{3}{10}} \psi_1 + e^{i\phi} \sqrt{\frac{7}{10}} \psi_2, \quad \phi \in \mathbb{R}.$$

Therefore, an energy measurement gives 10 eV with probability 3/10 and 30 eV with probability 7/10. The specified mean energy fixes these probabilities but does not fix the relative phase ϕ .

CSE • 2025 • 10 MARKS

Q4. Explain how the uncertainty in position is different from the uncertainty or inaccuracy of the measuring instruments.

INTRODUCTION:

The question asks for the distinction between the quantum-mechanical uncertainty in the position of a particle and the uncertainty introduced by an imperfect measuring instrument. Consider an ensemble of identically prepared particles in a fixed reference frame. Let the position observable be represented by the operator $\hat{x}$, and let the normalized state of each particle be $|\psi\rangle$. The intrinsic position uncertainty is defined statistically by

$$\Delta x = \sqrt{\langle (\hat{x} - \langle \hat{x} \rangle)^2 \rangle} = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2},$$

where $\langle \hat{x} \rangle = \langle \psi | \hat{x} | \psi \rangle$. Instrumental inaccuracy, on the other hand, is a property of the measurement apparatus and its calibration, resolution, stability, and interaction with the system.

SOLUTION:

In classical physics, a particle is ordinarily assumed to possess a definite position at every instant. If repeated measurements give different values, the spread is attributed to imperfect instruments, environmental disturbances, or lack of experimental control. In principle, one expects this spread to be reduced indefinitely by improving the apparatus.

Quantum-mechanical position uncertainty has a different origin. If the particle is described in the position representation by a normalized wavefunction $\psi(x)$, then

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1,$$

and the probability of finding it between x and $x + dx$ is

$$dP = |\psi(x)|^2 dx.$$

The expectation values are

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\psi(x)|^2 dx, \quad \langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx,$$

so that

$$\Delta x = \left[\int_{-\infty}^{\infty} (x - \langle x \rangle)^2 |\psi(x)|^2 dx \right]^{1/2}.$$

Thus, Δx measures the spread of the position probabilities inherent in the prepared quantum state. Even an ideal apparatus, assumed to have perfect resolution and calibration, generally produces a distribution of position outcomes when the experiment is repeated on identically prepared systems. This spread is not merely the observer's ignorance of a pre-existing exact value in the classical sense; it is part of the quantum description of the state.

The intrinsic character of this uncertainty is expressed by the position-momentum uncertainty relation. Since

$$[\hat{x}, \hat{p}_x] = i\hbar,$$

the Robertson relation gives

$$\Delta x \Delta p_x \geq \frac{1}{2} |\langle [\hat{x}, \hat{p}_x] \rangle| = \frac{\hbar}{2}.$$

Consequently, preparing a state with a smaller Δx necessarily requires a larger spread Δp_x . This limitation follows from the non-commuting structure of the observables and remains even for an ideal measuring instrument. It is therefore not a statement about defective experimental technique.

Instrumental uncertainty may instead be represented by a measurement error ϵ_x , determined, for example, by the detector's finite scale division, point-spread width, noise, parallax, calibration error, or systematic offset. It answers the question: how faithfully does the apparatus report the position outcome? Such errors may be classified as follows:

1. Random instrumental errors cause repeated readings to fluctuate and may often be reduced by better resolution, noise control, or statistical averaging.
2. Systematic errors, such as a zero error or incorrect calibration, shift the readings in a definite manner and can often be estimated and corrected by calibration.
3. Measurement disturbance arises when the measuring interaction changes the state of the particle. It is apparatus-dependent and should not be identified automatically with the intrinsic standard deviation Δx .

For an unbiased apparatus whose random error is statistically independent of the intrinsic position distribution, a simple model is

$$x_{\text{read}} = x + \delta x_{\text{inst}}, \quad \langle \delta x_{\text{inst}} \rangle = 0.$$

The variance of the recorded readings is then

$$\sigma_{\text{read}}^2 = (\Delta x)^2 + \epsilon_x^2,$$

where

$$\epsilon_x^2 = \langle (\delta x_{\text{inst}})^2 \rangle.$$

This relation shows that the experimentally observed spread may contain both intrinsic quantum uncertainty and instrumental imprecision. If the apparatus is progressively improved so that

$$\epsilon_x \longrightarrow 0,$$

then

$$\sigma_{\text{read}} \longrightarrow \Delta x,$$

not necessarily to zero. Hence the residual spread obtained with an ideal instrument is the position uncertainty of the quantum state.

Conversely, Δx can vanish only for an idealized exact position eigenstate. Such a state is non-normalizable over the whole real line and, consistently with $\Delta x \Delta p_x \geq \hbar/2$, would require an unbounded momentum spread. Any physically realizable localized wave packet therefore has a finite spatial spread.

The two concepts must also be distinguished operationally. The intrinsic Δx is inferred from the statistics of many measurements on an ensemble of systems prepared in the same state; it is not the uncertainty of a single scale reading. Instrumental inaccuracy characterizes the reliability of the apparatus and can be assessed through calibration and repeated measurements of known standards. Improving the instrument reduces ϵ_x , whereas changing the quantum preparation is required to change Δx .

CONCLUSION:

The uncertainty in position,

$$\Delta x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2},$$

is the intrinsic statistical spread of position outcomes for identically prepared quantum systems. It is constrained by

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}$$

and cannot, in general, be removed by constructing a more accurate apparatus.

Instrumental uncertainty ϵ_x is instead caused by finite resolution, noise, calibration errors, or other limitations of the measuring procedure. It is apparatus-dependent and can in principle be reduced or corrected. For independent unbiased instrumental error,

$$\sigma_{\text{read}}^2 = (\Delta x)^2 + \epsilon_x^2,$$

so an ideal instrument removes the instrumental contribution but still reveals the intrinsic quantum position distribution.

CSE • 2025 • 5 MARKS

Q5. Is it possible for a photon to transfer all its energy to a free electron? Give reasons.

INTRODUCTION:

The question asks whether an isolated free electron can completely absorb a photon,

$$\gamma + e^- \longrightarrow e^-,$$

while conserving both energy and momentum. Assume that the electron is initially at rest in the laboratory frame, with rest mass m , and that the incident photon has energy $E_\gamma = h\nu > 0$ and momentum magnitude $p_\gamma = E_\gamma/c$. After absorption, let the electron have momentum p and total relativistic energy E_e . No other particle or external body is available to carry energy or momentum.

SOLUTION:

If the photon were absorbed completely, conservation of momentum would require

$$p = p_\gamma = \frac{E_\gamma}{c}.$$

Conservation of energy would simultaneously require

$$mc^2 + E_\gamma = E_e.$$

For a free relativistic electron,

$$E_e^2 = p^2 c^2 + m^2 c^4.$$

Using the momentum-conservation result,

$$E_e^2 = \left(\frac{E_\gamma}{c}\right)^2 c^2 + m^2 c^4 = E_\gamma^2 + m^2 c^4.$$

On the other hand, energy conservation gives

$$E_e^2 = (mc^2 + E_\gamma)^2 = m^2 c^4 + E_\gamma^2 + 2mc^2 E_\gamma.$$

Equating these two expressions for E_e^2 yields

$$m^2 c^4 + E_\gamma^2 = m^2 c^4 + E_\gamma^2 + 2mc^2 E_\gamma,$$

and hence

$$2mc^2 E_\gamma = 0.$$

Since $m \neq 0$, this condition can be satisfied only for

$$E_\gamma = 0.$$

Thus, for any real photon with $E_\gamma > 0$, simultaneous conservation of energy and momentum is impossible.

The result is independent of the frame chosen. If the initial electron has four-momentum P^μ , the photon has four-momentum K^μ , and the final electron has four-momentum $P'^\mu = P^\mu + K^\mu$, then

$$P^2 = P'^2 = m^2 c^2, \quad K^2 = 0.$$

Therefore,

$$P'^2 = (P + K)^2 = P^2 + 2P \cdot K + K^2 = m^2 c^2 + 2P \cdot K.$$

For a future-directed massive electron and a real photon of positive energy, $P \cdot K > 0$; hence $P'^2 > m^2 c^2$, contradicting the required mass shell of a free electron.

Complete photon absorption can occur only if another object participates—for example, a bound atom, a nucleus, or a solid lattice—so that the additional system can take up the recoil momentum while conserving total energy.

CONCLUSION:

No non-zero-energy photon can transfer all its energy to a single isolated free electron. Complete absorption would violate simultaneous conservation of relativistic energy and momentum. A recoil partner or an external field is necessary to absorb the excess momentum; otherwise the photon can transfer only part of its energy, as in Compton scattering, and a scattered photon must remain.

IFoS • 2025 • 6+2=8 MARKS

Q6. A bullet of mass 0.025 kg is moving with a velocity of 600 m/s . The speed is measured with an accuracy of 0.02% . Find out the uncertainty in x . Also, comment on the result.

INTRODUCTION:

The speed accuracy determines the uncertainty in the bullet's momentum, which in turn limits the precision with which its position can be specified. We use the Heisenberg uncertainty relation

$$\Delta x \Delta p_x \geq \frac{\hbar}{2} = \frac{h}{4\pi}.$$

Here Δx and Δp_x are the position and momentum uncertainties along the direction of motion. The quoted percentage accuracy is interpreted as the uncertainty in the measured speed. Since the bullet's mass is treated as exactly known and its motion is non-relativistic,

$$p_x = mv, \quad \Delta p_x = m \Delta v.$$

The minimum possible position uncertainty corresponds to equality in the uncertainty relation.

SOLUTION:

The fractional uncertainty in speed is

$$\frac{\Delta v}{v} = \frac{0.02}{100} = 2.0 \times 10^{-4}.$$

Hence

$$\Delta v = v(2.0 \times 10^{-4}) = (600 \text{ m s}^{-1})(2.0 \times 10^{-4}) = 0.12 \text{ m s}^{-1}.$$

Therefore, the momentum uncertainty is

$$\begin{aligned} \Delta p_x &= m \Delta v \\ &= (0.025 \text{ kg})(0.12 \text{ m s}^{-1}) \\ &= 3.0 \times 10^{-3} \text{ kg m s}^{-1}. \end{aligned}$$

Using

$$\Delta x_{\min} = \frac{\hbar}{2\Delta p_x},$$

with

$$\hbar = 1.055 \times 10^{-34} \text{ J s},$$

we obtain

$$\begin{aligned} \Delta x_{\min} &= \frac{1.055 \times 10^{-34} \text{ J s}}{2(3.0 \times 10^{-3} \text{ kg m s}^{-1})} \\ &= \frac{1.055 \times 10^{-34} \text{ kg m}^2 \text{ s}^{-1}}{6.0 \times 10^{-3} \text{ kg m s}^{-1}} \\ &= 1.76 \times 10^{-32} \text{ m}. \end{aligned}$$

Equivalently, since the bullet momentum is

$$p = mv = (0.025)(600) = 15 \text{ kg m s}^{-1},$$

its relative momentum uncertainty is

$$\frac{\Delta p_x}{p} = \frac{3.0 \times 10^{-3}}{15} = 2.0 \times 10^{-4},$$

consistent with the stated relative speed uncertainty.

The calculated minimum uncertainty is extraordinarily smaller than the dimensions of a bullet and even much smaller than a nuclear length scale, of order 10^{-15} m . Thus the quantum-mechanical lower bound on its position uncertainty is experimentally and macroscopically negligible. The bullet can therefore be treated as following a well-defined classical trajectory, although it still formally obeys the uncertainty principle.

CONCLUSION:

The speed uncertainty and corresponding momentum uncertainty are

$$\Delta v = 0.12 \text{ m s}^{-1},$$

$$\Delta p_x = 3.0 \times 10^{-3} \text{ kg m s}^{-1}.$$

Consequently, the minimum position uncertainty along the direction of motion is

$$\Delta x_{\min} \approx 1.8 \times 10^{-32} \text{ m}.$$

This is a lower bound obtained by interpreting the quoted speed accuracy as Δv and taking the minimum-uncertainty case. It has no independent spatial direction; it is the scalar spread in position along the chosen direction of motion. Its extremely small magnitude shows why quantum uncertainty is imperceptible for a macroscopic bullet and why classical mechanics provides an excellent description of its motion.

CSE • 2024 • 10 MARKS

Q7. A particle limited to the x -axis has the wave function $\phi(x) = bx^2$ between $x = 0$ and $x = 2$; the wave function $\phi(x) = 0$ elsewhere.

(i) Find the probability that the particle can be found between $x = 1.0$ and $x = 1.5$. (ii) Find the expectation value $\langle x \rangle$ of the particle position.

INTRODUCTION:

The problem requires the probability of finding the particle in a specified interval and the expectation value of its position. In one-dimensional quantum mechanics, the probability density is

$$\rho(x) = |\phi(x)|^2,$$

and a physically admissible wave function must satisfy

$$\int_{-\infty}^{\infty} |\phi(x)|^2 dx = 1.$$

We take the positive x -direction as the coordinate direction and write

$$\phi(x) = \begin{cases} bx^2, & 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

The constant b may be complex, but only $|b|^2$ enters measurable probabilities. An overall phase is physically irrelevant, so b may be chosen real and positive after normalization. If x has dimensions of length, then $[\phi] = L^{-1/2}$ and consequently $[b] = L^{-5/2}$.

SOLUTION:

First normalize the wave function:

$$1 = \int_{-\infty}^{\infty} |\phi(x)|^2 dx = |b|^2 \int_0^2 x^4 dx.$$

Therefore,

$$1 = |b|^2 \left[\frac{x^5}{5} \right]_0^2 = |b|^2 \frac{2^5}{5} = |b|^2 \frac{32}{5},$$

so that

$$|b|^2 = \frac{5}{32}, \quad |b| = \sqrt{\frac{5}{32}}.$$

(i) The probability of finding the particle between $x = 1.0$ and $x = 1.5$ is

$$P(1.0 \leq x \leq 1.5) = \int_1^{1.5} |\phi(x)|^2 dx = |b|^2 \int_1^{1.5} x^4 dx.$$

Substituting the normalization constant,

$$P = \frac{5}{32} \left[\frac{x^5}{5} \right]_1^{1.5} = \frac{1}{32} \left[\left(\frac{3}{2} \right)^5 - 1 \right].$$

Since

$$\left(\frac{3}{2} \right)^5 = \frac{243}{32},$$

we obtain

$$P = \frac{1}{32} \left(\frac{243}{32} - 1 \right) = \frac{1}{32} \left(\frac{211}{32} \right) = \frac{211}{1024}.$$

Numerically,

$$P = \frac{211}{1024} \approx 0.2061,$$

or approximately 20.6%.

(ii) The expectation value of the particle position is

$$\langle x \rangle = \int_{-\infty}^{\infty} \phi^*(x) x \phi(x) dx = \int_0^2 x |\phi(x)|^2 dx.$$

Thus,

$$\langle x \rangle = |b|^2 \int_0^2 x^5 dx = \frac{5}{32} \left[\frac{x^6}{6} \right]_0^2.$$

Hence,

$$\langle x \rangle = \frac{5}{32} \frac{2^6}{6} = \frac{5}{32} \frac{64}{6} = \frac{5}{3}.$$

The result satisfies $0 < \langle x \rangle < 2$, as required because the probability density vanishes outside this interval. Moreover, since $|\phi(x)|^2 \propto x^4$, the probability is weighted toward larger x , so $\langle x \rangle = 5/3 > 1$ is physically reasonable.

CONCLUSION:

The normalized wave function, up to an arbitrary constant phase, is

$$\phi(x) = \begin{cases} \sqrt{\frac{5}{32}} x^2, & 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

The requested results are

$$P(1.0 \leq x \leq 1.5) = \frac{211}{1024} \approx 0.2061$$

and

$$\langle x \rangle = \frac{5}{3} \approx 1.67.$$

Thus, the probability is approximately 20.6%, and the mean position lies toward the upper end of the allowed interval because the probability density increases as x^4 . The value of $\langle x \rangle$ has the same position unit as the stated x -coordinates.

IFoS • 2024 • 8 MARKS

Q8. Determine the size of the hydrogen atom using uncertainty principle. Given that the potential energy of electron $V = \frac{-e^2}{4\pi\epsilon_0 a}$, where a is the distance of the electron from the nucleus.

INTRODUCTION:

The electron cannot remain at rest near the proton: localizing it within a region of characteristic radius a necessarily produces momentum uncertainty and hence kinetic energy. In the proton rest frame, neglecting proton recoil and relativistic effects, the electron has total energy

$$E = K + V, \quad V(a) = -\frac{e^2}{4\pi\epsilon_0 a}.$$

Let m_e be the electron mass. For localization over the distance a , the uncertainty principle gives the order-of-magnitude relation

$$\Delta x \Delta p \sim \hbar, \quad \Delta x \sim a,$$

so that

$$p \sim \Delta p \sim \frac{\hbar}{a}.$$

The uncertainty principle fixes the characteristic atomic length scale; numerical factors of order unity depend on the precise definition of the position spread. The conventional estimate $pa \sim \hbar$ yields the Bohr radius.

SOLUTION:

The kinetic energy associated with the momentum uncertainty is

$$K(a) \sim \frac{p^2}{2m_e} \sim \frac{\hbar^2}{2m_e a^2}.$$

Hence the energy of an electron localized at a characteristic distance a is

$$E(a) \sim \frac{\hbar^2}{2m_e a^2} - \frac{e^2}{4\pi\epsilon_0 a}.$$

The first term increases as a^{-2} when the electron is confined more strongly, whereas the attractive

Coulomb energy decreases as $-a^{-1}$. Their competition therefore permits a stable finite atomic size.

The equilibrium size is found by minimizing $E(a)$:

$$\frac{dE}{da} = -\frac{\hbar^2}{m_e a^3} + \frac{e^2}{4\pi\epsilon_0 a^2} = 0.$$

Thus

$$\frac{\hbar^2}{m_e a^3} = \frac{e^2}{4\pi\epsilon_0 a^2},$$

and therefore

$$a = \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2}.$$

This is the Bohr radius a_0 .

Using

$$\epsilon_0 = 8.854 \times 10^{-12} \text{ F m}^{-1}, \quad \hbar = 1.055 \times 10^{-34} \text{ J s},$$

$$m_e = 9.109 \times 10^{-31} \text{ kg}, \quad e = 1.602 \times 10^{-19} \text{ C},$$

one obtains

$$a_0 = \frac{4\pi(8.854 \times 10^{-12})(1.055 \times 10^{-34})^2}{(9.109 \times 10^{-31})(1.602 \times 10^{-19})^2} \text{ m} \\ \approx 5.29 \times 10^{-11} \text{ m}.$$

Equivalently,

$$a_0 \approx 0.529 \text{ \AA}.$$

The stationary point is a minimum because

$$\frac{d^2 E}{da^2} = \frac{3\hbar^2}{m_e a^4} - \frac{2e^2}{4\pi\epsilon_0 a^3}.$$

At $a = a_0$, the equilibrium condition gives

$$\frac{e^2}{4\pi\epsilon_0 a_0^3} = \frac{\hbar^2}{m_e a_0^4},$$

and hence

$$\left. \frac{d^2 E}{da^2} \right|_{a_0} = \frac{\hbar^2}{m_e a_0^4} > 0.$$

Thus the uncertainty-induced kinetic energy prevents collapse into the nucleus. Moreover, at the minimum,

$$\frac{e^2}{4\pi\epsilon_0 a_0} = \frac{\hbar^2}{m_e a_0^2} = 2K,$$

so $V = -2K$ and $E = K + V = -K < 0$, consistently describing a bound state.

CONCLUSION:

The uncertainty principle gives the characteristic size of the hydrogen atom as

$$a_0 = \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2} \approx 5.29 \times 10^{-11} \text{ m} = 0.529 \text{ \AA}.$$

This is the Bohr radius, interpreted as the characteristic electron-proton separation in the ground

state. The result follows in the non-relativistic, fixed-nucleus Coulomb approximation, with the uncertainty estimate $pa \sim \hbar$; replacing m_e by the electron-proton reduced mass gives the small finite-nuclear-mass correction.

CSE • 2023 • 15 MARKS

Q9. A particle constrained to move along x -axis in the domain $0 \leq x \leq L$ has a wave function $\psi(x) = \sin\left(\frac{n\pi x}{L}\right)$, where n is an integer. Normalize the wave function and evaluate the expectation value of momentum of the particle.

INTRODUCTION:

The required tasks are to determine the normalization constant and to calculate the expectation value of the momentum operator. Let

$$\psi_N(x) = A \sin\left(\frac{n\pi x}{L}\right), \quad 0 \leq x \leq L,$$

and $\psi_N(x) = 0$ outside this interval. Here $L > 0$, n must be a nonzero integer, and A may be chosen real and positive because an overall constant phase has no physical significance. The normalization condition and the momentum operator in the position representation are

$$\int_0^L |\psi_N(x)|^2 dx = 1, \quad \hat{p}_x = -i\hbar \frac{d}{dx}.$$

The positive x -direction is taken as the positive direction of momentum.

SOLUTION:

Applying the normalization condition,

$$1 = |A|^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx.$$

Using

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2},$$

the integral becomes

$$\begin{aligned} \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx &= \frac{1}{2} \int_0^L \left[1 - \cos\left(\frac{2n\pi x}{L}\right)\right] dx \\ &= \left[\frac{x}{2} - \frac{L}{4n\pi} \sin\left(\frac{2n\pi x}{L}\right)\right]_0^L \\ &= \frac{L}{2}, \end{aligned}$$

since $\sin(2n\pi) = 0$ for integer n . Hence

$$|A|^2 \frac{L}{2} = 1 \quad \Rightarrow \quad |A| = \sqrt{\frac{2}{L}}.$$

Choosing A real and positive gives

$$\psi_N(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right).$$

The expectation value of momentum is

$$\langle p_x \rangle = \int_0^L \psi_N^*(x) \left(-i\hbar \frac{d}{dx}\right) \psi_N(x) dx.$$

Since the normalized wave function is real,

$$\frac{d\psi_N}{dx} = \sqrt{\frac{2}{L}} \frac{n\pi}{L} \cos\left(\frac{n\pi x}{L}\right).$$

Therefore,

$$\begin{aligned} \langle p_x \rangle &= -i\hbar \frac{2n\pi}{L^2} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx \\ &= -i\hbar \frac{2n\pi}{L^2} \left[\frac{L}{2n\pi} \sin^2\left(\frac{n\pi x}{L}\right) \right]_0^L \\ &= -\frac{i\hbar}{L} [\sin^2(n\pi) - \sin^2(0)] \\ &= 0. \end{aligned}$$

Equivalently, for any real wave function vanishing at both boundaries,

$$\langle p_x \rangle = -\frac{i\hbar}{2} [\psi_N^2(x)]_0^L = 0.$$

Although the mean momentum vanishes, the state is not a momentum eigenstate. Indeed,

$$\sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2i} [e^{in\pi x/L} - e^{-in\pi x/L}],$$

so the standing wave contains equal contributions associated with opposite momenta $+\pi\hbar/L$ and $-\pi\hbar/L$; their contributions to the mean cancel.

CONCLUSION:

For $L > 0$ and any nonzero integer n , the normalized wave function, up to an arbitrary constant phase, is

$$\psi_N(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad 0 \leq x \leq L$$

and it vanishes outside the allowed interval. Its momentum expectation value is

$$\langle p_x \rangle = 0 \text{ kg m s}^{-1}.$$

Thus the particle has no preferred average direction of motion: the positive- and negative-momentum contributions are balanced. The case $n = 0$ is excluded because it gives the identically zero function, which cannot be normalized.

IFoS • 2023 • 8 MARKS

Q10. Find the condition at which de Broglie wavelength equals the Compton wavelength for a particle.

INTRODUCTION:

For a free particle of rest mass m , the de Broglie wavelength is

$$\lambda_{dB} = \frac{h}{p},$$

where p is the magnitude of its relativistic momentum. Its Compton wavelength is

$$\lambda_C = \frac{h}{mc}.$$

Since the required momentum may be relativistic, we use

$$p = \gamma mv, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}},$$

in the inertial frame in which the particle moves with speed v .

SOLUTION:

Equating the two wavelengths,

$$\lambda_{dB} = \lambda_C,$$

gives

$$\frac{h}{p} = \frac{h}{mc}.$$

Therefore,

$$p = mc.$$

Using $p = \gamma mv$,

$$\gamma mv = mc \implies \gamma v = c.$$

Writing $\beta = v/c$, this condition becomes

$$\frac{\beta}{\sqrt{1 - \beta^2}} = 1.$$

Hence,

$$\beta^2 = 1 - \beta^2, \quad 2\beta^2 = 1,$$

so that

$$\beta = \frac{1}{\sqrt{2}}, \quad v = \frac{c}{\sqrt{2}}.$$

The positive root is taken because v denotes the speed.

Equivalently, the total relativistic energy is

$$E = \sqrt{p^2 c^2 + m^2 c^4} = \sqrt{m^2 c^4 + m^2 c^4} = \sqrt{2} mc^2,$$

and therefore the kinetic energy is

$$K = E - mc^2 = (\sqrt{2} - 1)mc^2.$$

The non-relativistic substitution $p = mv$ would incorrectly give $v = c$, where the non-relativistic momentum formula is invalid. The relativistic result $v = c/\sqrt{2} < c$ is physically admissible.

CONCLUSION:

The de Broglie wavelength of a massive particle equals its Compton wavelength when

$$\boxed{p = mc}.$$

In terms of the particle speed and energy, the equivalent conditions are

$$\boxed{v = \frac{c}{\sqrt{2}}}, \quad \boxed{E = \sqrt{2} mc^2}, \quad \boxed{K = (\sqrt{2} - 1)mc^2}.$$



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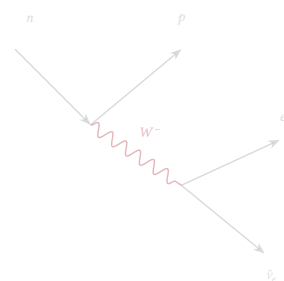
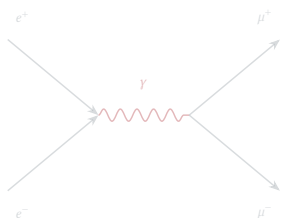
$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$E_n = -\frac{13.6}{n^2} \text{ eV}$$

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$$E^2 = p^2 c^2 + m^2 c^4$$

$$\partial_\mu j^\mu = 0$$

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Sample Preview extbar 10 Solved Questions

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CSE • 2026 • 10 MARKS

Q1. In a Stern–Gerlach experiment, the magnetic field varies with distance in the z -direction according to $\frac{dB_z}{dz} = 1.5 \text{ T/mm}$. The speed of silver atoms coming from the oven is 725 m/s . Silver atoms travel a distance of 3 cm through the magnet. Calculate the separation of the two beams of atoms when they leave the magnet. (4pt) [Given : The mass of a silver atom $= 1.8 \times 10^{-25} \text{ kg}$ and its magnetic moment is 1 Bohr magneton.]

INTRODUCTION:

The required quantity is the transverse separation of the two silver-atom beams at the exit of the magnet. Let the incident beam travel along the x -direction and let $+z$ be the direction in which B_z increases, so that

$$\frac{dB_z}{dz} > 0.$$

In an ideal Stern–Gerlach magnet, an atom with magnetic-moment component μ_z has magnetic potential energy

$$U = -\mu_z B_z$$

and therefore experiences the force

$$F_z = -\frac{\partial U}{\partial z} = \mu_z \frac{dB_z}{dz}.$$

For silver atoms the two allowed moment components are taken as

$$\mu_z = \pm \mu_B, \quad \mu_B = 9.27 \times 10^{-24} \text{ J T}^{-1}.$$

We assume a constant field gradient throughout the 3 cm magnet, negligible initial velocity in the z -direction, unchanged longitudinal speed, and negligible fringe-field and gravitational effects.

SOLUTION:

First convert the magnetic-field gradient to SI units:

$$\frac{dB_z}{dz} = 1.5 \text{ T mm}^{-1} = 1.5 \times 10^3 \text{ T m}^{-1}.$$

Thus the two magnetic-moment orientations experience equal and opposite forces,

$$F_{z,\pm} = \pm \mu_B \frac{dB_z}{dz},$$

whose common magnitude is

$$F = (9.27 \times 10^{-24} \text{ J T}^{-1})(1.5 \times 10^3 \text{ T m}^{-1}) = 1.39 \times 10^{-20} \text{ N}.$$

The corresponding transverse accelerations are

$$a_{z,\pm} = \frac{F_{z,\pm}}{m} = \pm \frac{\mu_B}{m} \frac{dB_z}{dz},$$

so their common magnitude is

$$a = \frac{1.39 \times 10^{-20} \text{ N}}{1.8 \times 10^{-25} \text{ kg}} = 7.73 \times 10^4 \text{ m s}^{-2}.$$

The time spent inside the magnet is determined by the longitudinal motion:

$$t = \frac{L}{v} = \frac{3.0 \times 10^{-2} \text{ m}}{725 \text{ m s}^{-1}} = 4.14 \times 10^{-5} \text{ s}.$$

Since the initial transverse velocity is zero, the displacement of each component is

$$z_{\pm} = \frac{1}{2} a_{z,\pm} t^2.$$

Hence the two displacements are equal in magnitude and opposite in direction. Their separation at the magnet exit is therefore

$$\Delta z = z_+ - z_- = at^2 = \frac{\mu_B}{m} \frac{dB_z}{dz} \left(\frac{L}{v} \right)^2.$$

Substituting the data,

$$\begin{aligned} \Delta z &= \frac{(9.27 \times 10^{-24} \text{ J T}^{-1})(1.5 \times 10^3 \text{ T m}^{-1})}{1.8 \times 10^{-25} \text{ kg}} \left(\frac{3.0 \times 10^{-2} \text{ m}}{725 \text{ m s}^{-1}} \right)^2 \\ &= (7.73 \times 10^4 \text{ m s}^{-2})(4.14 \times 10^{-5} \text{ s})^2 \\ &= 1.32 \times 10^{-4} \text{ m}. \end{aligned}$$

Equivalently, each beam is displaced from the undeflected axis by

$$|z_{\pm}| = \frac{\Delta z}{2} = 6.62 \times 10^{-5} \text{ m}.$$

The component with $\mu_z = +\mu_B$ moves toward $+z$, the direction of increasing field, whereas the component with $\mu_z = -\mu_B$ moves toward $-z$. The result also has the expected dependence $\Delta z \propto (dB_z/dz)L^2/v^2$: the splitting vanishes for zero field gradient and increases for a longer interaction time.

CONCLUSION:

The separation of the two silver-atom beams as they leave the magnet is

$$\Delta z = 1.3 \times 10^{-4} \text{ m} \approx 0.13 \text{ mm}.$$

The two beams are separated along the z -direction: the $\mu_z = +\mu_B$ component is deflected toward $+z$, while the $\mu_z = -\mu_B$ component is deflected toward $-z$. Each beam is approximately $6.6 \times 10^{-5} \text{ m}$ from the original beam axis. This result applies for a uniform field gradient, negligible initial transverse velocity and fringe fields, and approximately constant longitudinal speed; the opposite deflections demonstrate the two allowed orientations of the silver atom's magnetic moment.

CSE • 2026 • 8 MARKS

Q2. Why can the experimental observation of the Lamb shift not be explained by the Dirac theory?

INTRODUCTION:

The Lamb shift is the experimentally observed separation of atomic energy levels that the one-electron Dirac theory predicts to be degenerate, most notably the $2S_{1/2}$ and $2P_{1/2}$ levels of hydrogen. The question therefore requires a comparison between the Dirac spectrum in a static Coulomb field and the experimentally observed spectrum.

We assume an infinitely heavy, pointlike nucleus producing the Coulomb potential

$$V(r) = -\frac{Ze^2}{4\pi\epsilon_0 r}.$$

In the Dirac theory the electron is relativistic, but the electromagnetic field is treated as a prescribed classical field. Consequently, electron spin and relativistic effects are included, whereas the quantized electromagnetic vacuum, radiative self-energy, vacuum polarization, and nuclear recoil or finite-size corrections are absent.

SOLUTION:

The stationary Dirac equation for an electron in the nuclear Coulomb field is

$$[c \boldsymbol{\alpha} \cdot \mathbf{p} + \beta mc^2 + V(r)] \psi = E\psi.$$

Its bound-state energy eigenvalues may be written as

$$E_{nj} = mc^2 \left[1 + \frac{(Z\alpha)^2}{\left(n - j - \frac{1}{2} + \sqrt{\left(j + \frac{1}{2} \right)^2 - (Z\alpha)^2} \right)^2} \right]^{-1/2},$$

where n is the principal quantum number, j is the total electronic angular-momentum quantum number, and

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c}$$

is the fine-structure constant.

A decisive feature of this result is that the Coulomb-Dirac energy depends on n and j , but not separately on the orbital angular-momentum quantum number l . For the two states

$$2S_{1/2} : \quad n = 2, \quad l = 0, \quad j = \frac{1}{2},$$

and

$$2P_{1/2} : \quad n = 2, \quad l = 1, \quad j = \frac{1}{2},$$

the values of n and j are identical. Hence the Dirac formula gives

$$E_{2S_{1/2}}^{\text{Dirac}} = E_{2P_{1/2}}^{\text{Dirac}},$$

or

$$\Delta E_{\text{Dirac}} = E_{2S_{1/2}}^{\text{Dirac}} - E_{2P_{1/2}}^{\text{Dirac}} = 0.$$

This degeneracy is not merely a consequence of a low-order approximation. It is an exact property of the Dirac equation for an electron in an ideal $1/r$ Coulomb potential. The Dirac theory successfully accounts for the leading relativistic fine structure, including spin-orbit coupling and the relativistic correction to the kinetic energy, but these contributions still leave states with the same n and j degenerate.

Experimentally, however, the $2S_{1/2}$ level lies above the $2P_{1/2}$ level by a small but nonzero amount:

$$E_{2S_{1/2}} - E_{2P_{1/2}} > 0.$$

This observed separation is the Lamb shift. Since the uncorrected Dirac theory predicts the separation to be exactly zero, the observation cannot be explained within that theory.

The missing physics appears when the electromagnetic field is quantized. In quantum electrodynamics, an atomic electron interacts continually with vacuum electromagnetic fluctuations. The principal radiative corrections are:

1. The electron self-energy: the electron emits and reabsorbs virtual photons, thereby changing its bound-state energy.
2. Vacuum polarization: virtual electron-positron pairs modify the effective Coulomb interaction between the nucleus and the electron.

These corrections depend on the spatial form of the electron wavefunction and therefore distinguish states having the same n and j but different l . In particular, an S -state wavefunction is nonzero at the nucleus,

$$|\psi_{nS}(0)|^2 \neq 0,$$

whereas a P -state wavefunction vanishes there,

$$|\psi_{nP}(0)|^2 = 0.$$

Short-distance radiative modifications of the electron-nucleus interaction consequently affect the $2S_{1/2}$ state more strongly than the $2P_{1/2}$ state. The degeneracy of the Coulomb-Dirac spectrum is therefore removed.

Schematically, the physical energies have the form

$$E_{nlj} = E_{nj}^{\text{Dirac}} + \Delta E_{nlj}^{\text{self-energy}} + \Delta E_{nlj}^{\text{vacuum polarization}} + \dots,$$

so that

$$E_{2S_{1/2}} - E_{2P_{1/2}} = \Delta E_{2S_{1/2}}^{\text{QED}} - \Delta E_{2P_{1/2}}^{\text{QED}} \neq 0.$$

The experimentally observed sign, with the $2S_{1/2}$ level shifted upward relative to $2P_{1/2}$, is reproduced by the full radiative calculation.

CONCLUSION:

The Dirac equation in a static Coulomb potential predicts that energy depends only on n and j . It therefore gives

$$E_{2S_{1/2}}^{\text{Dirac}} = E_{2P_{1/2}}^{\text{Dirac}}, \quad \Delta E_{\text{Dirac}} = 0.$$

Experiment instead finds a nonzero splitting, with $2S_{1/2}$ above $2P_{1/2}$. Thus the Lamb shift lies beyond the one-particle Dirac theory and demonstrates the necessity of quantum-electrodynamic radiative corrections, chiefly electron self-energy and vacuum polarization.

CSE • 2026 • 8 MARKS

Q3. Calculate the number of permitted electrons in a sub-shell and a shell in an atom.

INTRODUCTION:

The maximum number of electrons permitted in an atomic state is determined by the quantum numbers and the Pauli exclusion principle: no two electrons in an atom can have the same set of four quantum numbers

$$(n, l, m_l, m_s).$$

Here $n = 1, 2, 3, \dots$ is the principal quantum number, $l = 0, 1, \dots, n-1$ is the orbital angular-momentum quantum number, $m_l = -l, -l+1, \dots, l$ is the magnetic quantum number, and $m_s = \pm \frac{1}{2}$ is the spin quantum number. A sub-shell is specified by fixed n and l , whereas a shell is specified only by fixed n .

SOLUTION:

For a sub-shell of orbital quantum number l , the allowed values of m_l are

$$-l, -l+1, \dots, 0, \dots, l-1, l.$$

Hence the number of distinct spatial orbitals in that sub-shell is

$$N_{\text{orb}} = l - (-l) + 1 = 2l + 1.$$

For each orbital, m_s can take the two values $+\frac{1}{2}$ and $-\frac{1}{2}$. Thus, by the Pauli exclusion principle, each orbital can accommodate at most two electrons. Therefore,

$$N_{\text{sub-shell}} = 2(2l + 1).$$

Consequently, the maximum occupancies of the s, p, d , and f sub-shells are respectively

Sub-shell	l	$2(2l + 1)$
s	0	2
p	1	6
d	2	10
f	3	14

For a shell with principal quantum number n , the allowed sub-shells have

$$l = 0, 1, 2, \dots, n-1.$$

The maximum number of electrons in the shell is therefore the sum of the capacities of all its sub-shells:

$$N_{\text{shell}} = \sum_{l=0}^{n-1} 2(2l + 1).$$

Using

$$\sum_{l=0}^{n-1} (2l + 1) = 2 \frac{n(n-1)}{2} + n = n^2,$$

we obtain

$$N_{\text{shell}} = 2n^2.$$

For example, the $n = 1, 2, 3$, and 4 shells can contain at most

$$2, \quad 8, \quad 18, \quad 32$$

electrons, respectively. For $n = 1$, only $l = 0$ is permitted, so the shell capacity $2n^2 = 2$ correctly agrees with the $1s$ sub-shell capacity.

CONCLUSION:

A sub-shell with orbital quantum number l contains $2l + 1$ orbitals and can accommodate a maximum of

$$2(2l + 1) \text{ electrons}.$$

A shell with principal quantum number n , containing all sub-shells $l = 0$ to $n-1$, can accommodate a maximum of

$$2n^2 \text{ electrons}.$$

These maximum occupancies follow directly from the allowed quantum-number states and the Pauli exclusion principle.

CSE • 2026 • 12 MARKS

Q4. Explain how Pauli's exclusion principle helps in determining the electronic configuration in a many-electron system.

INTRODUCTION:

An electronic configuration specifies how the electrons of an atom are distributed among its orbitals. In a many-electron atom, a one-electron state is labelled by four quantum numbers:

$$(n, l, m_l, m_s),$$

where n is the principal quantum number, l the orbital angular-momentum quantum number, m_l its magnetic quantum number, and $m_s = \pm \frac{1}{2}$ the spin projection.

Electrons are identical spin- $\frac{1}{2}$ fermions, so their total many-electron wavefunction must be antisymmetric under the interchange of any two electrons. This requirement gives Pauli's exclusion principle:

No two electrons in an atom can have the same set of all four quantum numbers.

The principle determines the maximum allowed occupation of every orbital and subshell. The actual order of filling is additionally governed by orbital energies, while the distribution among degenerate orbitals is governed by Hund's rules.

SOLUTION:

For a specified spatial orbital (n, l, m_l) , the only remaining quantum number is the spin projection. Since an electron can have only

$$m_s = +\frac{1}{2} \quad \text{or} \quad m_s = -\frac{1}{2},$$

at most two electrons can occupy that spatial orbital. If two electrons occupy it, their spins must be opposite:

$$(n, l, m_l, +\frac{1}{2}), \quad (n, l, m_l, -\frac{1}{2}).$$

A third electron would necessarily repeat one of these two complete sets of quantum numbers and is therefore forbidden.

For a subshell with orbital quantum number l ,

$$m_l = -l, -l+1, \dots, l,$$

so there are $2l+1$ distinct spatial orbitals. Each orbital can contain two electrons; hence the maximum capacity of the subshell is

$$N_l = 2(2l+1).$$

Thus,

subshell	l	maximum number of electrons
s	0	2
p	1	6
d	2	10
f	3	14

For a shell of principal quantum number n , the allowed values are

$$l = 0, 1, \dots, n-1.$$

Therefore its maximum electronic capacity is

$$N_n = \sum_{l=0}^{n-1} 2(2l+1) = 2 \sum_{l=0}^{n-1} (2l+1) = 2n^2,$$

because the sum of the first n odd integers is n^2 . Hence the K , L , and M shells can accommodate at most 2, 8, and 18 electrons, respectively.

To construct the ground-state configuration of an atom with atomic number Z , its Z electrons are placed successively in the available one-electron states:

1. Orbitals are filled approximately in increasing order of energy, according to the Aufbau principle.
2. Pauli's principle restricts each orbital to at most two electrons with opposite spins.
3. When several orbitals of the same subshell are degenerate, Hund's first rule places electrons singly with parallel spins before pairing occurs, thereby reducing electron-electron repulsion and lowering the energy.

For example, carbon has $Z = 6$. Filling the lowest-energy orbitals gives

$$1s^2 2s^2 2p^2.$$

The $1s$ and $2s$ orbitals each contain two electrons of opposite spin, as required by Pauli's principle. The two $2p$ electrons occupy two different p -orbitals with parallel spins in the ground state:

$$2p_x^1 2p_y^1 2p_z^0,$$

up to an equivalent choice of the degenerate p -orbitals. Pauli's principle forbids placing more than two electrons in any one of these orbitals; Hund's rule determines the energetically preferred distribution before pairing.

The wavefunction basis of this restriction can be seen from a two-electron antisymmetric state,

$$\Psi(1, 2) = \frac{1}{\sqrt{2}} [\phi_a(1)\phi_b(2) - \phi_a(2)\phi_b(1)].$$

If the two electrons were assigned the same spin-orbital, $\phi_a = \phi_b = \phi$, then

$$\Psi(1, 2) = \frac{1}{\sqrt{2}} [\phi(1)\phi(2) - \phi(2)\phi(1)] = 0.$$

Thus a physical many-electron state cannot contain two electrons in the same spin-orbital. This is why Pauli's principle fixes the allowed occupancies and gives the characteristic shell structure underlying the periodic table.

CONCLUSION:

Pauli's exclusion principle determines electronic configurations by allowing each spin-orbital to be occupied by at most one electron, or equivalently each spatial orbital by at most two electrons with opposite spins. Consequently, a subshell with quantum number l holds at most $2(2l + 1)$ electrons and a shell with principal quantum number n holds at most $2n^2$ electrons. Together with the orbital-energy ordering and Hund's rules, this restriction uniquely specifies the allowed ground-state distribution of electrons and accounts for atomic shell structure, valency, and the periodic recurrence of chemical properties.

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Q5. The Stern-Gerlach experiment is a landmark experiment in quantum mechanics. Discuss about the most important findings of this experiment.

INTRODUCTION:

The Stern-Gerlach experiment tests whether the component of an atomic magnetic moment along a chosen direction can vary continuously, as expected classically, or can assume only discrete quantum values. A collimated beam of neutral atoms is passed through a strongly inhomogeneous magnetic field and the spatial distribution of the emerging atoms is recorded.

Choose a laboratory frame in which the atomic beam travels along $+x$. Near the beam axis, let the magnetic field be predominantly along z ,

$$\mathbf{B} \simeq B_z(z) \hat{\mathbf{z}}, \quad G \equiv \frac{\partial B_z}{\partial z} > 0.$$

Thus increasing z is the direction of increasing field magnitude. This convention will be used for all force and path assignments.

The atoms are assumed to move independently in vacuum. Their centre-of-mass motion may be treated semiclassically, whereas their internal angular momentum is treated quantum mechanically. The field direction is assumed to vary sufficiently slowly along an atomic trajectory that the magnetic-moment eigenstate follows the local field direction. Nuclear magnetic moments and other small corrections are neglected in comparison with the electronic magnetic moment. Neutral atoms are used so that there is no ordinary Lorentz force $\mathbf{q}\mathbf{v} \times \mathbf{B}$.

The governing interaction between an atomic magnetic moment $\boldsymbol{\mu}$ and the field is

$$H_{\text{int}} = -\boldsymbol{\mu} \cdot \mathbf{B}.$$

The experiment therefore converts the allowed eigenvalues of the magnetic-moment component into distinguishable spatial deflections.

SOLUTION:

For a moment eigenstate having a definite component μ_z along the local field, its magnetic interaction energy is

$$E(z) = -\mu_z B_z(z).$$

The corresponding force follows from the spatial derivative of the energy:

$$F_z = -\frac{\partial E}{\partial z} = \mu_z \frac{\partial B_z}{\partial z} = \mu_z G.$$

Consequently, a uniform magnetic field may produce a torque or a Zeeman energy splitting, but it cannot separate the beam spatially because $G = 0$. The inhomogeneity of the field is essential.

Classically, the magnetic moments in an unpolarized beam could be oriented at any angle θ to the z -axis. Their components would then form a continuous range,

$$\mu_z = \mu \cos \theta, \quad -\mu \leq \mu_z \leq \mu,$$

and hence

$$-\mu G \leq F_z \leq \mu G.$$

The detector should therefore show a continuously broadened band rather than sharply separated components.

Quantum mechanically, an atom with total electronic angular momentum $\mathbf{J}$ has

$$J^2 = J(J+1)\hbar^2, \quad J_z = m_J \hbar,$$

where

$$m_J = -J, -J+1, \dots, J.$$

For an electronic magnetic moment,

$$\boldsymbol{\mu} = -g_J \mu_B \frac{\mathbf{J}}{\hbar}, \quad \mu_B = \frac{e\hbar}{2m_e},$$

where g_J is the Lande factor and μ_B is the Bohr magneton. Hence

$$\mu_z = -g_J \mu_B m_J$$

and the allowed forces are

$$F_z = -g_J \mu_B m_J G.$$

There are only $2J+1$ possible values of m_J , so an ideal beam separates into $2J+1$ discrete components.

Stern and Gerlach used silver atoms. The ground-state electronic configuration is

$$[\text{Kr}] 4d^{10} 5s^1.$$

The closed shells make no net contribution to the electronic angular momentum. For the single 5s electron,

$$L = 0, \quad S = \frac{1}{2}, \quad J = \frac{1}{2},$$

so the silver ground state is a two-state system with

$$m_J = +\frac{1}{2} \quad \text{or} \quad m_J = -\frac{1}{2}.$$

Since $L = 0$, the Lande factor is approximately

$$g_J \approx 2.$$

Therefore,

$$\begin{aligned} m_J = +\frac{1}{2} : \quad & \mu_z = -\mu_B, \quad F_z = -\mu_B G, \\ m_J = -\frac{1}{2} : \quad & \mu_z = +\mu_B, \quad F_z = +\mu_B G. \end{aligned}$$

With the convention $G > 0$, the $m_J = +\frac{1}{2}$ state is deflected toward decreasing z , while the $m_J = -\frac{1}{2}$ state is deflected toward increasing z . This reversal between angular momentum and magnetic-moment directions occurs because the electron has negative charge. In the limit $G \rightarrow 0$, both forces vanish, as required for a uniform field.

The most important findings and implications are as follows.

1. The observed silver beam divided into two distinct components instead of forming the continuous band predicted by classical orientation theory. This supplied direct evidence for *space quantization*: only discrete projections of atomic angular momentum and magnetic moment are obtained along the measurement axis.

The quantization axis is not a permanently preferred direction in space. It is selected by the orientation of the Stern–Gerlach magnetic field.

2. The twofold splitting is explained in modern quantum mechanics by the intrinsic spin of the unpaired 5s electron:

$$S = \frac{1}{2}, \quad S_z = \pm \frac{\hbar}{2}.$$

The experiment was performed before electron spin had been introduced, so it was originally interpreted as evidence for quantized atomic angular momentum. Its modern interpretation establishes the characteristic two-valuedness of a spin- $\frac{1}{2}$ system.

3. The result shows that intrinsic spin is not an ordinary classical rotation of an extended body. For any chosen measurement axis, a spin- $\frac{1}{2}$ particle yields only two component values, even though a classical angular-momentum vector could have a continuous range of projections.

4. A Stern–Gerlach apparatus acts as a spin analyzer. If a general incident spin state is written as

$$|\psi\rangle = a|+z\rangle + b|-z\rangle, \quad |a|^2 + |b|^2 = 1,$$

where

$$S_z|\pm z\rangle = \pm \frac{\hbar}{2}|\pm z\rangle,$$

then the two separated paths occur with probabilities

$$P(+z) = |a|^2, \quad P(-z) = |b|^2.$$

Here the labels $+z$ and $-z$ denote spin eigenvalues, not spatial deflection directions. For $G > 0$,

$$|+z\rangle \longrightarrow \text{path toward decreasing } z$$

$$|-z\rangle \longrightarrow \text{path toward increasing } z.$$

If one output path is selected and the other is blocked, the apparatus also prepares the atoms in a definite spin eigenstate.

5. Successive Stern–Gerlach measurements demonstrate the incompatibility of spin components along different axes. An atom prepared in $|+z\rangle$ gives the same definite result when measured again along z . However,

$$|+z\rangle = \frac{1}{\sqrt{2}}(|+x\rangle + |-x\rangle),$$

so a subsequent measurement along x produces the two S_x values with equal probabilities:

$$P(+x) = P(-x) = \frac{1}{2}.$$

This follows from the angular-momentum commutation relation

$$[S_x, S_z] = -i\hbar S_y \neq 0.$$

Thus S_x and S_z cannot simultaneously possess sharp values.

6. The experiment demonstrates how a quantum internal degree of freedom becomes correlated with a macroscopic spatial path. The inhomogeneous field produces the association

$$\begin{aligned} m_J = +\frac{1}{2} & \longleftrightarrow F_z = -\mu_B G \\ m_J = -\frac{1}{2} & \longleftrightarrow F_z = +\mu_B G. \end{aligned}$$

This state-dependent separation is the basic principle behind spin filtering, atomic-beam magnetic-resonance techniques and many quantum-state measurements.

Sharp resolution of the two components requires the field gradient and interaction time to be sufficiently large, while the incident beam must be sufficiently well collimated that the separation exceeds the spread caused by the initial transverse velocities. The atomic magnetic state must also remain correlated with the local field direction during passage through the magnet. These conditions distinguish the quantum splitting from ordinary geometrical broadening of the beam.

CONCLUSION:

The decisive observation of the Stern–Gerlach experiment is that a neutral silver-atom beam separates into two discrete components rather than a continuous distribution. For the silver ground state,

$$\begin{aligned} J &= S = \frac{1}{2} \\ m_J &= \pm \frac{1}{2} \\ \mu_z &= -g_J \mu_B m_J \\ g_J &\approx 2 \end{aligned}$$

where μ_B is measured in J T^{-1} . With

$$G = \frac{\partial B_z}{\partial z} > 0,$$

the two force eigenvalues, measured in newtons, are

$$m_J = +\frac{1}{2} : F_z = -\mu_B G \quad \text{toward decreasing } z,$$

and

$$m_J = -\frac{1}{2} : F_z = +\mu_B G \quad \text{toward increasing } z.$$

Provided that the beam is adequately collimated, the magnetic-field gradient is sufficiently strong, and the internal state remains correlated with the local field, the experiment establishes discrete angular-momentum projections and the two-valued spin- $\frac{1}{2}$ degree of freedom. Its broader inference is that quantum measurement is probabilistic and axis-dependent: a Stern–Gerlach apparatus can analyse and prepare spin states, while spin components along different axes are incompatible because their operators do not commute. Atomic angular momentum is therefore intrinsically quantum mechanical and cannot be represented by a classical continuous distribution of magnetic-moment orientations.

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Q6. Draw the normal Zeeman pattern for $^1F_3 - ^1D_2$ transition.

INTRODUCTION:

The magnetic field is taken along the $+z$ -direction,

$$\mathbf{B} = B\hat{z}, \quad B > 0,$$

and the weak-field Zeeman regime is assumed, so that the spin-orbit coupling remains intact and L, S, J, M are good quantum numbers. The first-order magnetic interaction is

$$H_Z = \frac{\mu_B}{\hbar} (\mathbf{L} + 2\mathbf{S}) \cdot \mathbf{B},$$

giving the level shift

$$\Delta E = \mu_B g_J B M,$$

where

$$g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}.$$

Let M_u and M_l denote the magnetic quantum numbers of the upper and lower levels. The sign convention used for a component is

$$q = M_u - M_l.$$

For an electric-dipole transition,

$$q = 0, \pm 1.$$

The $q = 0$ component is denoted by π , while $q = +1$ and $q = -1$ are denoted by σ_+ and σ_- , respectively.

The required pattern is obtained by finding the magnetic sublevels, the allowed transitions between them, and the corresponding frequency and polarization of each distinct component.

SOLUTION:

For both terms the multiplicity is $2S + 1 = 1$, hence

$$S = 0.$$

Therefore $J = L$, and the Lande factor reduces to

$$g_J = 1 + \frac{J(J+1) - L(L+1)}{2J(J+1)} = 1.$$

Thus

$$g_u = g(^1F_3) = 1, \quad g_l = g(^1D_2) = 1,$$

which is the defining condition for a normal Zeeman pattern.

The upper level has

$$J_u = 3, \quad M_u = -3, -2, -1, 0, 1, 2, 3,$$

and therefore splits into seven equally spaced sublevels:

$$E_u(M_u) = E_u^{(0)} + \mu_B B M_u.$$

The lower level has

$$J_l = 2, \quad M_l = -2, -1, 0, 1, 2,$$

and splits into five equally spaced sublevels:

$$E_l(M_l) = E_l^{(0)} + \mu_B B M_l.$$

If

$$\nu_0 = \frac{E_u^{(0)} - E_l^{(0)}}{h}$$

is the zero-field transition frequency, an allowed Zeeman transition has frequency

$$\begin{aligned} \nu(M_u, M_l) &= \frac{E_u(M_u) - E_l(M_l)}{h} \\ &= \nu_0 + \frac{\mu_B B}{h} (M_u - M_l). \end{aligned}$$

Defining

$$\nu_L = \frac{\mu_B B}{h},$$

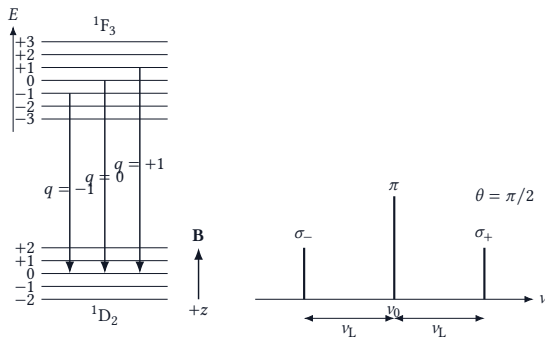
this becomes

$$\nu = \nu_0 + q\nu_L, \quad q = 0, \pm 1.$$

Although there are fifteen allowed magnetic-sublevel transitions, all transitions with the same q have the same frequency. They therefore form only three resolved spectral components:

q	allowed (M_u, M_l)	frequency	component
-1	$(-3, -2), (-2, -1), (-1, 0), (0, 1), (1, 2)$	$\nu_0 - \nu_L$	σ_-
0	$(-2, -2), (-1, -1), (0, 0), (1, 1), (2, 2)$	ν_0	π
+1	$(-1, -2), (0, -1), (1, 0), (2, 1), (3, 2)$	$\nu_0 + \nu_L$	σ_+

The sublevel splitting, representative transitions, and the spectrum for observation perpendicular to $\mathbf{B}$ are shown below. The central line is drawn with twice the height of either side line because the transverse intensity ratio is $2 : 1 : 1$.



For an equally populated upper level, rotational symmetry and the Wigner-Eckart theorem imply equal total angle-integrated strengths for the three groups $q = -1, 0, +1$. Their observed intensities nevertheless depend on the angle θ between the line of sight and $\mathbf{B}$. The angular factors are proportional to

$$I_\pi(\theta) \propto \sin^2 \theta, \quad I_{\sigma_\pm}(\theta) \propto \frac{1 + \cos^2 \theta}{2}.$$

Consequently, for transverse observation, $\theta = \pi/2$,

$$I_\pi : I_{\sigma_+} : I_{\sigma_-} = 2 : 1 : 1.$$

The π component is then linearly polarized parallel to $\mathbf{B}$, whereas the two σ components are linearly polarized perpendicular to $\mathbf{B}$.

For longitudinal observation, $\theta = 0$,

$$I_\pi = 0, \quad I_{\sigma_+} = I_{\sigma_-},$$

and the two σ components are oppositely circularly polarized. Their handedness is interchanged if the direction of observation is reversed.

As $B \rightarrow 0$, $\nu_L \rightarrow 0$, so all three components merge continuously into the original line at ν_0 , as required.

CONCLUSION:

The $^1F_3 - ^1D_2$ transition gives a normal Zeeman triplet because both singlet levels have

$$g_u = g_l = 1.$$

With $\mathbf{B} = B\hat{z}$, $B > 0$, and $q = M_u - M_l$, the three components are

$$\begin{aligned} \nu_{\sigma_-} &= \nu_0 - \frac{\mu_B B}{h} \\ \nu_\pi &= \nu_0 \\ \nu_{\sigma_+} &= \nu_0 + \frac{\mu_B B}{h}. \end{aligned}$$

Here $\mu_B B/h$ is in hertz. The adjacent frequency separation is

$$\Delta\nu = \frac{\mu_B B}{h},$$

and the separation of the two σ components is

$$\nu_{\sigma_+} - \nu_{\sigma_-} = \frac{2\mu_B B}{h}.$$

Thus σ_- is displaced toward lower frequency and longer wavelength, while σ_+ is displaced toward higher frequency and shorter wavelength. For transverse observation the polarization directions are

$$\pi \parallel \mathbf{B}, \quad \sigma_\pm \perp \mathbf{B},$$

with intensity ratio

$$I_\pi : I_{\sigma_+} : I_{\sigma_-} = 2 : 1 : 1.$$

For longitudinal observation the π component is absent and the two σ components are oppositely circularly polarized. These results apply in the weak-field Zeeman regime in which J remains a good quantum number.

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Q7. $n_1 = 6$, $l_1 = 3$, $s_1 = \frac{1}{2}$, $n_2 = 5$, $l_2 = 1$, $s_2 = \frac{1}{2}$. The quantum numbers of two electrons in a two-valence electron atom are $n_1 = 6$, $l_1 = 3$, $s_1 = \frac{1}{2}$, $n_2 = 5$, $l_2 = 1$, $s_2 = \frac{1}{2}$. Assuming L - S coupling, find the possible values of L and J .

INTRODUCTION:

In L - S coupling, the individual orbital angular momenta first combine to form

$$\mathbf{L} = \mathbf{l}_1 + \mathbf{l}_2,$$

and the two electron spins combine to form

$$\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2.$$

The resultant orbital and spin angular momenta then combine according to

$$\mathbf{J} = \mathbf{L} + \mathbf{S}.$$

Here L , S , and J denote the corresponding non-negative angular-momentum quantum numbers, with magnitudes

$$|\mathbf{L}| = \hbar\sqrt{L(L+1)}$$

$$|\mathbf{S}| = \hbar\sqrt{S(S+1)}$$

$$|\mathbf{J}| = \hbar\sqrt{J(J+1)}.$$

An arbitrary fixed z -axis may be used as the quantization axis; the allowed resultant quantum numbers do not depend on this choice. The electrons occupy the distinct orbitals $6f$ and $5p$, so they are non-equivalent electrons. Consequently, no additional Pauli exclusion restriction removes any of the angular-momentum couplings obtained below.

SOLUTION:

For the addition of two angular momenta l_1 and l_2 , the allowed values of the resultant orbital quantum number are

$$L = |l_1 - l_2|, |l_1 - l_2| + 1, \dots, l_1 + l_2.$$

Substituting $l_1 = 3$ and $l_2 = 1$,

$$L = |3 - 1|, |3 - 1| + 1, \dots, 3 + 1,$$

and hence

$$L = 2, 3, 4.$$

These values are conventionally denoted by the spectroscopic letters

$$L = 2 \leftrightarrow D, \quad L = 3 \leftrightarrow F, \quad L = 4 \leftrightarrow G.$$

The two spin quantum numbers are $s_1 = s_2 = \frac{1}{2}$. Their resultant spin is therefore

$$S = |s_1 - s_2|, \dots, s_1 + s_2 = \left| \frac{1}{2} - \frac{1}{2} \right|, \dots, \left(\frac{1}{2} + \frac{1}{2} \right),$$

so that

$$S = 0, 1.$$

Thus both singlet states, with multiplicity $2S + 1 = 1$, and triplet states, with multiplicity $2S + 1 = 3$, are possible.

For each pair (L, S) , the total angular-momentum quantum number satisfies

$$J = |L - S|, |L - S| + 1, \dots, L + S.$$

For the singlet states, $S = 0$, so

$$J = |L - 0| = L.$$

Therefore,

L	S	J
2	0	2
3	0	3
4	0	4

and the corresponding singlet terms are

$$^1D_2, \quad ^1F_3, \quad ^1G_4.$$

For the triplet states, $S = 1$, so for every allowed $L \geq 1$,

$$J = L - 1, L, L + 1.$$

Consequently,

L	S	J
2	1	1, 2, 3
3	1	2, 3, 4
4	1	3, 4, 5

and the corresponding triplet terms are

$$^3D_1, ^3D_2, ^3D_3,$$

$$^3F_2, ^3F_3, ^3F_4,$$

$$^3G_3, ^3G_4, ^3G_5.$$

The extreme values provide a useful angular-momentum check: the largest possible total angular momentum is

$$J_{\max} = l_1 + l_2 + s_1 + s_2 = 3 + 1 + \frac{1}{2} + \frac{1}{2} = 5,$$

which agrees with the largest value $J = 5$ obtained for $L = 4, S = 1$. The smallest value is $J = 1$, obtained from the antiparallel coupling of $L = 2$ and $S = 1$.

CONCLUSION:

The allowed resultant orbital angular-momentum quantum numbers are

$$L = 2, 3, 4.$$

Including both allowed total-spin values $S = 0$ and $S = 1$, the possible J -values associated with each L are

$$\begin{array}{llll} L = 2 : & J = 2 & (S = 0), & J = 1, 2, 3 & (S = 1), \\ L = 3 : & J = 3 & (S = 0), & J = 2, 3, 4 & (S = 1), \\ L = 4 : & J = 4 & (S = 0), & J = 3, 4, 5 & (S = 1). \end{array}$$

Thus the distinct possible numerical values of the total angular-momentum quantum number are

$$J = 1, 2, 3, 4, 5,$$

with the allowed L - S - J associations specified above. These results apply in the assumed L - S coupling regime for the non-equivalent $6f$ and $5p$ electrons.

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Q8. Find out the difference in frequencies of Lyman-alpha line in hydrogen and deuterium atoms.

INTRODUCTION:

The Lyman- α line is produced by the transition

$$n = 2 \longrightarrow n = 1.$$

Hydrogen and deuterium have identical nuclear charge $+e$, but their nuclei have different masses. Consequently, their electron-nucleus reduced masses, and hence their Rydberg constants and transition frequencies, are slightly different.

Let m_e , m_p , and m_d denote the electron, proton, and deuteron masses, respectively. The reduced mass of an electron bound to a nucleus of mass M is

$$\mu_M = \frac{m_e M}{m_e + M}.$$

Nuclear recoil is included through this reduced mass. Fine structure, Lamb shift, and nuclear-size corrections are neglected.

SOLUTION:

For a hydrogen-like atom of nuclear charge $Z = 1$, the energy of the n th level is

$$E_n = -hcR_\infty \frac{\mu_M}{m_e} \frac{1}{n^2},$$

where R_∞ is the Rydberg constant for an infinitely massive nucleus.

Thus, for the Lyman- α transition,

$$h\nu_M = E_2 - E_1 = hcR_\infty \frac{\mu_M}{m_e} \left(1 - \frac{1}{2^2}\right),$$

so that

$$\nu_M = \frac{3}{4} cR_\infty \frac{\mu_M}{m_e}.$$

For ordinary hydrogen,

$$\frac{\mu_H}{m_e} = \frac{m_p}{m_e + m_p} = \frac{1}{1 + m_e/m_p},$$

whereas for deuterium,

$$\frac{\mu_D}{m_e} = \frac{m_d}{m_e + m_d} = \frac{1}{1 + m_e/m_d}.$$

Since $m_d > m_p$, one has $\mu_D > \mu_H$, and therefore the Lyman- α frequency of deuterium is higher. The frequency difference is

$$\Delta\nu \equiv \nu_D - \nu_H = \frac{3}{4} cR_\infty \left[\frac{1}{1 + m_e/m_d} - \frac{1}{1 + m_e/m_p} \right].$$

Using

$$\begin{aligned} \frac{m_p}{m_e} &\approx 1836.15 \\ \frac{m_d}{m_e} &\approx 3670.48 \end{aligned}$$

$$cR_\infty \approx 3.28984 \times 10^{15} \text{ s}^{-1}$$

gives

$$\Delta\nu = \frac{3}{4} (3.28984 \times 10^{15}) \left[\frac{1}{1 + 1/3670.48} - \frac{1}{1 + 1/1836.15} \right] \text{ Hz}.$$

The dimensionless reduced-mass difference is approximately

$$\frac{1}{1 + 1/3670.48} - \frac{1}{1 + 1/1836.15} \approx 2.72 \times 10^{-4}.$$

Therefore,

$$\Delta\nu \approx (2.46738 \times 10^{15})(2.72 \times 10^{-4}) \text{ Hz} \approx 6.7 \times 10^{11} \text{ Hz}.$$

The positive sign is physically consistent: the heavier deuteron recoils less than the proton, giving deuterium a larger reduced mass, a larger binding-energy separation, and hence a higher transition frequency.

CONCLUSION:

The difference between the Lyman- α frequencies of deuterium and hydrogen is

$$\nu_D - \nu_H \approx 6.7 \times 10^{11} \text{ Hz}.$$

Thus, the deuterium Lyman- α line has the higher frequency and, equivalently, the shorter wavelength.

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Q9. Why were silver atoms used in Stern-Gerlach experiment? Also, write the importance of this experiment.

INTRODUCTION:

The Stern-Gerlach experiment examines the behaviour of a beam of neutral atoms in a non-uniform magnetic field. If the field is taken along the z -direction, the force on an atom having magnetic-moment component μ_z is

$$F_z = \mu_z \frac{\partial B_z}{\partial z}.$$

Thus, different allowed values of μ_z produce different transverse deflections. Silver is especially suitable because its electronic structure makes the observed magnetic moment simple to interpret.

SOLUTION:

The electronic configuration of silver, $Z = 47$, is

$$\text{Ag} : [\text{Kr}] 4d^{10}5s^1.$$

The filled inner shells, including the $4d^{10}$ subshell, have paired electrons. Their orbital and spin angular momenta cancel, so they make no resultant contribution:

$$L_{\text{closed}} = 0, \quad S_{\text{closed}} = 0.$$

The single outer $5s$ electron has

$$l = 0, \quad s = \frac{1}{2}, \quad j = \frac{1}{2},$$

and hence the ground state is

$$^2S_{1/2}.$$

Since $l = 0$, the valence electron has no orbital angular momentum. The magnetic moment of the atom is therefore essentially the spin magnetic moment of this one unpaired electron. Its component along the magnetic field can take only two values because

$$m_j = \pm \frac{1}{2}.$$

Using

$$\mu_z = -g_J \mu_B m_j, \quad g_J \approx 2,$$

one obtains

$$\mu_z \approx \mp \mu_B,$$

where

$$\mu_B = \frac{e\hbar}{2m_e}$$

is the Bohr magneton. Consequently,

$$F_z = \mu_z \frac{\partial B_z}{\partial z}$$

has two opposite values, and the silver-atom beam separates into two components.

Silver atoms also have practical experimental advantages: a well-collimated neutral atomic beam can be produced by heating silver in an oven, and the atoms can be detected from the visible deposit formed on a collecting plate. Their neutrality prevents the ordinary Lorentz force on electric charge from obscuring the magnetic-moment deflection.

Classically, atomic magnetic moments could have arbitrary orientations. A classical beam would therefore be expected to form a continuous distribution of deflections. Instead, Stern and Gerlach observed two sharply separated components. The experiment consequently established the following points:

1. It provided direct evidence for *space quantization*: the component of atomic angular momentum, and hence of magnetic moment, along a chosen axis can assume only discrete values.
 2. The twofold splitting of the silver beam was later understood as evidence for the two allowed projections of electron spin,
- $$m_s = +\frac{1}{2}, \quad m_s = -\frac{1}{2}.$$
3. It demonstrated that measurement of an angular-momentum component gives discrete quantum outcomes rather than a continuous range of classical values.
 4. It supplied important experimental support for quantum mechanics and for the association between angular momentum and magnetic moment.
 5. It introduced the principle of separating quantum states by their magnetic moments, which underlies atomic-beam magnetic-resonance methods and later techniques for manipulating and measuring spin states.

Historically, the experiment was initially interpreted in terms of orbital space quantization because electron spin had not yet been proposed. The modern interpretation of the two silver components is based on the spin of the single $5s$ electron.

CONCLUSION:

Silver was used because its configuration $[\text{Kr}] 4d^{10}5s^1$ contains closed shells with zero resultant angular momentum and one unpaired s -electron with $l = 0$ and $j = \frac{1}{2}$. Its magnetic moment is therefore essentially a single-electron spin moment with two allowed field

components, approximately $\mu_z = \pm\mu_B$, producing two opposite deflections in a non-uniform magnetic field. The experiment established discrete space quantization, gave decisive evidence for the two-valued projection of electron spin, contradicted the classical prediction of continuous orientations, and became a foundational demonstration of quantum-state measurement and magnetic-state separation.

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Q10. Write the electronic configurations for carbon (C), nitrogen (N) and oxygen (O) atoms, and then derive their ground states.

INTRODUCTION:

The atomic numbers are $Z_C = 6$, $Z_N = 7$, and $Z_O = 8$. Their electronic configurations follow from the Aufbau principle, the Pauli exclusion principle, and Hund's rules.

For these atoms, the closed subshells $1s^2 2s^2$ have

$$L = 0, \quad S = 0,$$

and therefore do not affect the resultant term symbol. It is sufficient to couple the electrons in the open $2p^n$ subshell. For a p -electron,

$$l = 1, \quad m_l = +1, 0, -1, \quad s = \frac{1}{2}.$$

In Russell–Saunders coupling, an atomic level is denoted by

$$^{2S+1}L_J,$$

where S is the total spin, $L = 0, 1, 2, \dots$ is represented by $S, P, D, \dots$, and

$$J = |L - S|, |L - S| + 1, \dots, L + S.$$

Odd parity is indicated by a superscript $^\circ$.

Hund's rules select the ground level as follows: maximize S ; for that value of S , maximize L ; and then choose the minimum J for a less-than-half-filled subshell and the maximum J for a more-than-half-filled subshell. The $2p$ subshell is half-filled at $2p^3$.

SOLUTION:

- (a) For carbon, $Z = 6$, so the electronic configuration is

$$\boxed{\text{C} : 1s^2 2s^2 2p^2}.$$

The two $2p$ -electrons occupy different orbitals with parallel spins:

$$\begin{array}{c|ccc} m_l & +1 & 0 & -1 \\ \hline 2p^2 & \uparrow & \uparrow & \end{array}$$

Thus

$$M_S = \frac{1}{2} + \frac{1}{2} = 1,$$

which is the maximum possible value and therefore gives

$$S = 1, \quad 2S + 1 = 3.$$

For two parallel-spin electrons, the Pauli principle requires different m_l values. The largest possible orbital projection is consequently

$$M_L = 1 + 0 = 1,$$

so the corresponding maximum- M_L state has

$$L = 1 \quad (P).$$

Hence Hund's first two rules select the term

$$^3P.$$

Equivalently, the complete allowed terms of a p^2 configuration are

$$^3P, \quad ^1D, \quad ^1S,$$

and the triplet term lies lowest because it has the largest spin.

For $L = 1$ and $S = 1$,

$$J = 0, 1, 2.$$

Since $2p^2$ is less than half-filled, Hund's third rule selects the smallest value,

$$J = |L - S| = 0.$$

The parity is even because

$$(-1)^{2l} = (-1)^2 = +1.$$

Therefore the carbon ground state is

$$\boxed{\text{C} : 1s^2 2s^2 2p^2 \ ^3P_0}.$$

- (b) For nitrogen, $Z = 7$, so the electronic configuration is

$$\boxed{\text{N} : 1s^2 2s^2 2p^3}.$$

To maximize the total spin, the three $2p$ -electrons occupy all three p -orbitals with parallel spins:

$$\begin{array}{c|ccc} m_l & +1 & 0 & -1 \\ \hline 2p^3 & \uparrow & \uparrow & \uparrow \end{array}$$

Therefore

$$M_S = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2},$$

and hence

$$S = \frac{3}{2}, \quad 2S + 1 = 4.$$

The orbital projection of this unique maximum-spin arrangement is

$$M_L = (+1) + 0 + (-1) = 0.$$

Since all three available m_l orbitals must be singly occupied in the maximum-spin state, no state with $S = 3/2$ can have $M_L \neq 0$. Thus

$$L = 0 \quad (S).$$

Hund's rules therefore select

$$^4S.$$

The other terms of p^3 , such as ^2D and ^2P , have smaller spin and lie higher.

For $L = 0$ and $S = 3/2$, only one value of J is possible:

$$J = \frac{3}{2}.$$

The parity is odd because

$$(-1)^{3l} = (-1)^3 = -1.$$

Consequently the nitrogen ground state is

$$\boxed{\text{N} : 1s^2 2s^2 2p^3 \ ^4S_{3/2}^o}.$$

- (c) For oxygen, $Z = 8$, so the electronic configuration is

$$\boxed{\text{O} : 1s^2 2s^2 2p^4}.$$

Hund's first rule gives three parallel-spin electrons in the three different p -orbitals; the fourth electron must pair with opposite spin:

$$\frac{m_l}{2p^4} \begin{array}{|c|c|c|} \hline +1 & 0 & -1 \\ \hline \uparrow\downarrow & \uparrow & \uparrow \\ \hline \end{array}$$

For this maximum-spin arrangement,

$$M_S = \left(\frac{1}{2} - \frac{1}{2}\right) + \frac{1}{2} + \frac{1}{2} = 1,$$

so

$$S = 1, \quad 2S + 1 = 3.$$

Choosing the paired orbital at $m_l = +1$ gives the largest orbital projection compatible with $S = 1$:

$$M_L = (+1) + (+1) + 0 + (-1) = 1.$$

Therefore

$$L = 1 \quad (P),$$

and Hund's first two rules select the term

$$^3P.$$

The same result follows from the hole description: a p^4 configuration contains two holes in the otherwise closed p^6 subshell and therefore has the same allowed L and S terms as p^2 :

$$^3P, \quad ^1D, \quad ^1S.$$

For $L = 1$ and $S = 1$,

$$J = 0, 1, 2.$$

Since $2p^4$ is more than half-filled, Hund's third rule selects the largest value,

$$J = L + S = 2.$$

The parity is even:

$$(-1)^{4l} = (-1)^4 = +1.$$

Hence the oxygen ground state is

$$\boxed{\text{O} : 1s^2 2s^2 2p^4 \ ^3P_2}.$$

The contrast between carbon and oxygen provides a useful check: p^2 and p^4 have the same lowest term ^3P , but their ground J -values are opposite members of the multiplet because p^2 is less than half-filled whereas p^4 is more than half-filled.

CONCLUSION:

Under Russell–Saunders coupling for isolated light atoms, the electronic configurations and ground-state term symbols are

C : $1s^2 2s^2 2p^2$,	ground state $^3P_0$,
N : $1s^2 2s^2 2p^3$,	ground state $^4S_{3/2}^o$,
O : $1s^2 2s^2 2p^4$,	ground state $^3P_2$.

Thus carbon and oxygen possess triplet- P ground terms but different ground J -levels, while the exactly half-filled $2p^3$ subshell of nitrogen produces a spin-aligned, orbitally non-degenerate quartet- S ground state of odd parity.



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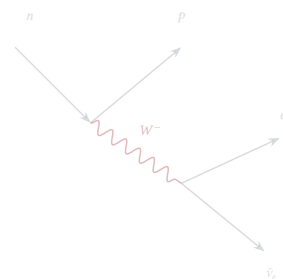
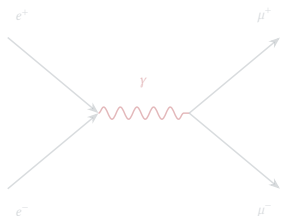
$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$E_n = -\frac{13.6}{n^2} \text{ eV}$$

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$$\partial_\mu j^\mu = 0$$

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Sample Preview extbar 10 Solved Questions

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8	What is the minimum energy required to break a ^4_2He nucleus into free protons and neutrons? [Given, $m_H = 1 \cdot 007825$ amu, $m_n = 1 \cdot 008665$ amu, $m_e = 0 \cdot 00055$ amu and $m_{\text{He}} = 4 \cdot 002603$ amu]	15	2025	CSE	13
9	Consider a uranium nucleus ($_{92}\text{U}^{236}$) breaking up spontaneously into two equal parts. Estimate the reduction of electrostatic energy of the nucleus considering uniform charge distribution. [Assume that nuclear radius is $1 \cdot 2 \times 10^{-13} A^{1/3}$ cm]	15	2025	CSE	14
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CSE • 2026 • 10 MARKS

Q1. Are the nucleons inside the nucleus governed by the laws of quantum physics? Justify your answer.

INTRODUCTION:

Yes. Protons and neutrons inside a nucleus are microscopic particles confined to a region of linear dimension of order a few femtometres. Their motion must therefore be described by quantum mechanics rather than by classical trajectories. This can be justified quantitatively by comparing the nucleon's de Broglie wavelength with the nuclear size and is also confirmed by characteristic nuclear phenomena such as discrete energy levels, spin and parity, shell structure, tunnelling, and the Pauli exclusion principle.

Let a nucleon of mass m_N be localized within a nucleus of radius

$$R \simeq r_0 A^{1/3}, \quad r_0 \simeq 1.2 \text{ fm},$$

where A is the mass number and $1 \text{ fm} = 10^{-15} \text{ m}$.

SOLUTION:

Localization of a nucleon to a distance of order R implies, from the Heisenberg uncertainty principle,

$$\Delta r \Delta p \gtrsim \frac{\hbar}{2}.$$

Taking $\Delta r \sim R$, the characteristic momentum is at least of order

$$p \sim \frac{\hbar}{R}.$$

Consequently, the corresponding reduced de Broglie wavelength is

$$\frac{\lambda}{2\pi} = \frac{\hbar}{p} \sim R,$$

or equivalently,

$$\lambda \sim 2\pi R.$$

Thus the matter-wave length scale of a confined nucleon is comparable to, or larger than, the dimensions of the nucleus. The nucleon cannot be represented as a localized classical particle following a definite orbit; its wave character is essential throughout the nuclear volume.

The same conclusion follows from the characteristic confinement energy. In the non-relativistic approximation,

$$E_{\text{conf}} \sim \frac{p^2}{2m_N} \sim \frac{\hbar^2}{2m_N R^2}.$$

For a representative nuclear length $R \sim 2 \text{ fm}$, using

$$\hbar c \simeq 197 \text{ MeV fm}, \quad m_N c^2 \simeq 939 \text{ MeV},$$

one obtains

$$E_{\text{conf}} \sim \frac{(\hbar c)^2}{2(m_N c^2)R^2} = \frac{(197 \text{ MeV fm})^2}{2(939 \text{ MeV})(2 \text{ fm})^2} \simeq 5.2 \text{ MeV}.$$

This energy scale is comparable to typical nuclear excitation and binding-energy scales, so quantum confinement is not a small correction to classical motion. It determines the nuclear states themselves. Moreover,

$$\frac{pc}{m_N c^2} \sim \frac{\hbar c}{R m_N c^2} \simeq \frac{197}{2 \times 939} \simeq 0.10,$$

so this illustrative estimate is consistent with a predominantly non-relativistic quantum treatment, although relativistic and subnucleonic corrections may be required for greater precision.

Several observed properties of nuclei provide direct evidence:

1. Nuclei possess discrete energy levels. Their emission and absorption of γ -rays occur at definite energies

$$E_\gamma = E_i - E_f,$$

as expected for transitions between quantized stationary states.

2. Protons and neutrons are spin- $\frac{1}{2}$ fermions. Identical nucleons obey the Pauli exclusion principle, so they fill nuclear quantum states rather than all collapsing into the lowest state. This produces Fermi motion even in the nuclear ground state.
3. Nuclear states have definite angular momentum and parity. The observed shell structure and the enhanced stability of nuclei having magic numbers are consequences of quantized single-particle states in an effective nuclear potential.
4. Processes such as α -decay occur by quantum tunnelling through the Coulomb barrier. Classically, an α -particle whose energy is below the barrier could not escape, whereas quantum mechanics gives a finite penetration probability.
5. A nucleus is described by a many-nucleon wavefunction,

$$\Psi(\mathbf{r}_1, \dots, \mathbf{r}_A; \sigma_1, \dots, \sigma_A),$$

which evolves according to the quantum Hamiltonian,

$$\hat{H}\Psi = E\Psi$$

for a stationary state. The wavefunction must have the appropriate antisymmetry under interchange of identical protons or identical neutrons.

The uncertainty-principle estimate also gives the correct classical limit: if the confining length R were made very large compared with the reduced de Broglie wavelength, then $\hbar/(pR) \ll 1$, quantum interference would become less important, and a classical description could emerge. In an actual nucleus, however, $\hbar/(pR)$ is of order unity, so that limit is not applicable.

CONCLUSION:

The nucleons inside a nucleus are necessarily governed by quantum physics. Their de Broglie wavelength is comparable to the nuclear dimensions, and confinement produces momenta and energies of nuclear magnitude. Hence nucleons do not follow definite classical trajectories; they occupy quantized many-body states characterized by energy, angular momentum, spin and parity. Discrete spectra, nuclear shell structure, Fermi motion, the Pauli exclusion principle and quantum tunnelling provide experimental manifestations of this quantum behaviour.

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Q2. Binding energy and rest mass energy of a two-nucleon bound state are denoted by B and Mc^2 , respectively, where c is the speed of light. Calculate the minimum energy of a photon required to dissociate this bound state in terms of B and Mc^2 .

INTRODUCTION:

Consider the bound state initially at rest in the laboratory frame. Let m_1 and m_2 be the rest masses of the two free nucleons. By definition of the binding energy,

$$B = (m_1 + m_2 - M)c^2$$

so that

$$(m_1 + m_2)c^2 = Mc^2 + B.$$

The photon is taken to propagate along the positive z -direction. Four-momenta are written as

$$p^\mu = \left(\frac{E}{c}, \mathbf{p}\right)$$

with metric signature $(+, -, -, -)$, so that $p^\mu p_\mu = m^2 c^2$. Energy and momentum conservation are most conveniently imposed through the invariant mass of the complete initial system. At threshold, the two outgoing nucleons have zero relative momentum in their centre-of-momentum frame, which minimizes their total invariant energy.

SOLUTION:

If E_γ is the incident photon energy, the four-momenta of the stationary target and the photon are

$$P^\mu = (Mc, \mathbf{0}), \quad k^\mu = \left(\frac{E_\gamma}{c}, 0, 0, \frac{E_\gamma}{c}\right).$$

Since the photon is massless,

$$k^2 = 0,$$

and

$$P \cdot k = (Mc) \left(\frac{E_\gamma}{c}\right) = ME_\gamma.$$

Therefore, the squared total initial four-momentum is

$$(P + k)^2 = P^2 + 2P \cdot k + k^2 = M^2 c^2 + 2ME_\gamma.$$

In the centre-of-momentum frame of the final nucleons, their momenta are equal and opposite:

$$\mathbf{p}_1^* = \mathbf{q}, \quad \mathbf{p}_2^* = -\mathbf{q}.$$

Their total energy in that frame is

$$E_f^*(q) = \sqrt{m_1^2 c^4 + q^2 c^2} + \sqrt{m_2^2 c^4 + q^2 c^2}.$$

This is minimized at $q = 0$. Hence, at the dissociation threshold,

$$E_{f,\text{th}}^* = (m_1 + m_2)c^2,$$

and the squared final total four-momentum is

$$(P_f)^2 = (m_1 + m_2)^2 c^2.$$

Four-momentum conservation gives $P + k = P_f$, so at threshold

$$M^2 c^2 + 2ME_{\gamma,\text{min}} = (m_1 + m_2)^2 c^2.$$

Multiplying by c^2 and using

$$(m_1 + m_2)c^2 = Mc^2 + B,$$

we obtain

$$M^2 c^4 + 2Mc^2 E_{\gamma,\text{min}} = (Mc^2 + B)^2.$$

Thus,

$$\begin{aligned} 2Mc^2 E_{\gamma,\text{min}} &= (Mc^2 + B)^2 - M^2 c^4 \\ &= 2Mc^2 B + B^2, \end{aligned}$$

which yields

$$E_{\gamma,\text{min}} = B + \frac{B^2}{2Mc^2} = B \left(1 + \frac{B}{2Mc^2}\right).$$

The threshold exceeds B because a photon carries momentum E_γ/c . Since the target is initially at rest, the final two-nucleon system must carry this momentum in the positive z -direction even when its internal relative kinetic energy is zero. For weak binding, $B \ll Mc^2$,

$$E_{\gamma,\text{min}} = B \left(1 + \frac{B}{2Mc^2}\right) \simeq B,$$

with the small positive recoil correction $B^2/(2Mc^2)$.

CONCLUSION:

For an isolated two-nucleon bound state of rest energy Mc^2 , initially at rest, the minimum photon energy required for dissociation is

$$E_{\gamma, \min} = B \left(1 + \frac{B}{2Mc^2} \right).$$

Here $E_{\gamma, \min}$, B , and Mc^2 are all measured in energy units. At threshold the two nucleons have zero relative momentum in their centre-of-momentum frame, while their total laboratory momentum is directed along the incident photon.

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Q3. A nucleus of rest mass M is initially in an excited state whose energy is ΔE above its ground state. The nucleus emits a γ -ray of energy $h\nu$ and makes a transition to its ground state. Calculate the fractional change in energy for the nucleus.

INTRODUCTION:

The emitted photon carries momentum

$$p_{\gamma} = \frac{h\nu}{c}.$$

Since the initial nucleus is at rest, conservation of momentum requires the ground-state nucleus to recoil with equal and opposite momentum. Consequently, the photon energy is slightly smaller than the excitation energy ΔE .

Let M denote the rest mass of the ground-state nucleus. The initial energy is then

$$E_i = Mc^2 + \Delta E.$$

If $E_{\gamma} = h\nu$ is the photon energy, the recoiling nucleus has momentum magnitude $p = E_{\gamma}/c$ and relativistic energy

$$E_N = \sqrt{M^2c^4 + p^2c^2}.$$

SOLUTION:

Energy and momentum conservation give

$$Mc^2 + \Delta E = E_{\gamma} + \sqrt{M^2c^4 + E_{\gamma}^2}.$$

Therefore,

$$Mc^2 + \Delta E - E_{\gamma} = \sqrt{M^2c^4 + E_{\gamma}^2}.$$

Squaring both sides,

$$(Mc^2 + \Delta E)^2 - 2E_{\gamma}(Mc^2 + \Delta E) + E_{\gamma}^2 = M^2c^4 + E_{\gamma}^2.$$

After cancellation,

$$2Mc^2\Delta E + (\Delta E)^2 = 2E_{\gamma}(Mc^2 + \Delta E),$$

so that

$$E_{\gamma} = h\nu = \frac{\Delta E(2Mc^2 + \Delta E)}{2(Mc^2 + \Delta E)}.$$

Equivalently,

$$h\nu = \Delta E \left(1 - \frac{\Delta E}{2(Mc^2 + \Delta E)} \right).$$

The energy retained by the nucleus as recoil kinetic energy is

$$T_R = \Delta E - h\nu = \frac{(\Delta E)^2}{2(Mc^2 + \Delta E)}.$$

Hence the fractional change, i.e. the fraction of the transition energy going into nuclear recoil, is

$$\frac{\Delta E - h\nu}{\Delta E} = \frac{\Delta E}{2(Mc^2 + \Delta E)}.$$

For ordinary nuclear transitions,

$$\Delta E \ll Mc^2,$$

and therefore

$$\frac{\Delta E - h\nu}{\Delta E} \simeq \frac{\Delta E}{2Mc^2}.$$

In the same approximation,

$$T_R \simeq \frac{(\Delta E)^2}{2Mc^2}, \quad h\nu \simeq \Delta E \left(1 - \frac{\Delta E}{2Mc^2} \right).$$

This agrees with the non-relativistic recoil expression

$$T_R = \frac{p^2}{2M} = \frac{(h\nu)^2}{2Mc^2} \simeq \frac{(\Delta E)^2}{2Mc^2}.$$

CONCLUSION:

The photon energy is reduced below the excitation energy because the ground-state nucleus must recoil. The fractional energy change is

$$\frac{\Delta E - h\nu}{\Delta E} = \frac{\Delta E}{2(Mc^2 + \Delta E)} \simeq \frac{\Delta E}{2Mc^2} \quad (\Delta E \ll Mc^2).$$

Thus the nucleus receives the recoil energy

$$T_R \simeq \frac{(\Delta E)^2}{2Mc^2}.$$

in the direction opposite to the emitted γ -ray, while

$$h\nu \simeq \Delta E \left(1 - \frac{\Delta E}{2Mc^2} \right).$$

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Q4. By considering the three-dimensional harmonic oscillator potential, explain the energy levels of nucleons inside the nucleus. Also, derive the zero-point energy of nucleons. Why did this assumption fail to explain the existence of nuclei with higher magic number?

INTRODUCTION:

In the independent-particle shell model, each nucleon is assumed to move in an average, spherically symmetric potential produced by all the other nucleons. Near the centre of a smooth potential well, its leading radial dependence is approximately quadratic. We therefore take

$$V(r) = -V_0 + \frac{1}{2}m\omega^2 r^2,$$

where m is the nucleon mass, ω is the oscillator frequency, and $-V_0$ is the value of the mean potential at its minimum. The coordinate $\mathbf{r}$ is measured from the nuclear centre, and the single-particle energy is measured relative to the zero chosen outside the potential. Thus, the excitation energy above the bottom of the well is independent of V_0 .

The assumptions are that the nucleons move independently, the mean field is central and identical in form for all nucleons of a given species, and residual interactions, Coulomb corrections and spin-orbit coupling are initially neglected. Protons and neutrons fill their respective levels separately according to the Pauli exclusion principle. Each spatial state can contain two identical nucleons because the nucleon has spin $s = \frac{1}{2}$.

SOLUTION:

1. The three-dimensional oscillator spectrum

The single-particle Hamiltonian is

$$H = -V_0 + \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2 r^2.$$

Since $r^2 = x^2 + y^2 + z^2$ and $\mathbf{p}^2 = p_x^2 + p_y^2 + p_z^2$, it separates into three one-dimensional oscillators:

$$H = -V_0 + \sum_{i=x,y,z} \left(\frac{p_i^2}{2m} + \frac{1}{2}m\omega^2 x_i^2 \right).$$

The energy eigenvalues are consequently

$$\epsilon_{n_x n_y n_z} = -V_0 + \hbar\omega \left(n_x + n_y + n_z + \frac{3}{2} \right) \\ n_x, n_y, n_z = 0, 1, 2, \dots$$

Introducing the principal oscillator quantum number

$$N = n_x + n_y + n_z,$$

one obtains

$$\epsilon_N = -V_0 + \left(N + \frac{3}{2} \right) \hbar\omega.$$

Thus all states with the same value of N are degenerate in the pure oscillator model. In spherical coordinates,

$$N = 2n_r + l,$$

where $n_r = 0, 1, 2, \dots$ is the radial quantum number and l is the orbital angular-momentum quantum number. Hence

$$\epsilon_{n_r l} = -V_0 + \left(2n_r + l + \frac{3}{2} \right) \hbar\omega.$$

For a given N , the allowed values are

$$l = N, N-2, N-4, \dots,$$

ending at $l = 0$ or $l = 1$. The parity of every state in the N -th oscillator shell is

$$\pi = (-1)^l = (-1)^N.$$

The number of non-negative integer triples satisfying $n_x + n_y + n_z = N$ is

$$g_N^{(\text{spatial})} = \binom{N+2}{2} = \frac{(N+1)(N+2)}{2}.$$

Including the two possible spin projections, the capacity of the N -th shell for protons or neutrons separately is

$$g_N = 2g_N^{(\text{spatial})} = (N+1)(N+2).$$

Therefore the shell capacities and cumulative occupation numbers are

N	orbitals in the shell	g_N	$\sum_{n=0}^N g_n$
0	1s	2	2
1	1p	6	8
2	1d, 2s	12	20
3	1f, 2p	20	40
4	1g, 2d, 3s	30	70
5	1h, 2f, 3p	42	112

Indeed,

$$\sum_{n=0}^N (n+1)(n+2) = \frac{(N+1)(N+2)(N+3)}{3}.$$

The pure three-dimensional harmonic oscillator therefore predicts the shell-closure sequence

$$\boxed{2, 8, 20, 40, 70, 112, \dots}$$

for neutron number or proton number separately. It correctly accounts for the first three observed magic numbers 2, 8, and 20.

2. Derivation of the zero-point energy

For each Cartesian coordinate, define the annihilation and creation operators

$$a_i = \sqrt{\frac{m\omega}{2\hbar}} x_i + \frac{i}{\sqrt{2m\hbar\omega}} p_i$$

$$a_i^\dagger = \sqrt{\frac{m\omega}{2\hbar}} x_i - \frac{i}{\sqrt{2m\hbar\omega}} p_i.$$

Using $[x_i, p_j] = i\hbar\delta_{ij}$, one finds

$$[a_i, a_j^\dagger] = \delta_{ij}$$

and

$$\frac{p_i^2}{2m} + \frac{1}{2}m\omega^2 x_i^2 = \hbar\omega \left(a_i^\dagger a_i + \frac{1}{2} \right).$$

Consequently,

$$H + V_0 = \hbar\omega \sum_{i=x,y,z} \left(a_i^\dagger a_i + \frac{1}{2} \right).$$

Since the eigenvalues of $a_i^\dagger a_i$ are the non-negative integers n_i , the smallest possible oscillator energy occurs for

$$n_x = n_y = n_z = 0.$$

It is

$$E_{zp} = \hbar\omega \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) = \boxed{\frac{3}{2}\hbar\omega}.$$

Thus the lowest single-particle energy, including the depth of the nuclear mean field, is

$$\boxed{\epsilon_0 = -V_0 + \frac{3}{2}\hbar\omega}.$$

The quantity $\frac{3}{2}\hbar\omega$ is the zero-point energy above the bottom of the well. It cannot be removed while retaining the oscillator potential because simultaneous vanishing of all three coordinate and momentum uncertainties would violate the uncertainty principle.

For A independent occupied single-particle states, the oscillator contribution to the total energy is

$$E_{osc} = \sum_{i=1}^A \left(N_i + \frac{3}{2} \right) \hbar\omega = \hbar\omega \sum_{i=1}^A N_i + \frac{3A}{2} \hbar\omega.$$

The Pauli principle prevents all identical nucleons from occupying $N_i = 0$, so the total energy also contains the shell-filling contribution $\hbar\omega \sum_i N_i$. A commonly used phenomenological nuclear estimate is

$$\hbar\omega \simeq 41A^{-1/3} \text{ MeV},$$

which gives the oscillator level spacing and makes the single-nucleon zero-point energy approximately

$$E_{zp} \simeq 61.5A^{-1/3} \text{ MeV}.$$

3. Failure for the higher magic numbers

The experimentally established higher magic numbers are

$$28, \quad 50, \quad 82, \quad 126,$$

whereas the pure oscillator gives 40, 70, 112, ... The failure follows from the excessive degeneracy of the harmonic oscillator. Its energy depends only on

$$N = 2n_r + l$$

and is independent of both l and the total angular momentum

$$\mathbf{j} = \mathbf{l} + \mathbf{s}.$$

Hence the two states

$$j = l + \frac{1}{2}, \quad j = l - \frac{1}{2}$$

are predicted to have the same energy.

In an actual nucleus there is a strong spin-orbit interaction,

$$V_{ls}(r) \mathbf{l} \cdot \mathbf{s},$$

and

$$\mathbf{l} \cdot \mathbf{s} = \frac{1}{2} [j(j+1) - l(l+1) - s(s+1)] \hbar^2.$$

For $s = \frac{1}{2}$,

$$\mathbf{l} \cdot \mathbf{s} = \begin{cases} \frac{l}{2} \hbar^2, & j = l + \frac{1}{2}, \\ -\frac{l+1}{2} \hbar^2, & j = l - \frac{1}{2}. \end{cases}$$

The nuclear spin-orbit interaction has the observed ordering in which the aligned state $j =$

$l + \frac{1}{2}$ is lowered substantially relative to the $j = l - \frac{1}{2}$ state. The splitting grows with l , so high- l orbitals may be displaced across the gap between adjacent oscillator shells.

For example, immediately above the closure at 20, the $1f_{7/2}$ level is pulled down. Its degeneracy is

$$2j + 1 = 2\left(\frac{7}{2}\right) + 1 = 8.$$

Filling this level therefore gives

$$20 + 8 = 28,$$

producing the observed magic number 28 instead of the oscillator closure at 40. Analogous rearrangements involving the $1g_{9/2}$, $1h_{11/2}$, and $1i_{13/2}$ orbitals lead to the closures at 50, 82, and 126, respectively.

The finite nuclear surface also makes the realistic mean field closer to a Woods–Saxon potential than to an oscillator at large r , thereby removing part of the accidental N -shell degeneracy. Nevertheless, the strong spin-orbit interaction is the decisive ingredient required to reproduce the higher magic numbers.

CONCLUSION:

For a nucleon moving in the spherical harmonic-oscillator mean field,

$$V(r) = -V_0 + \frac{1}{2}m\omega^2 r^2,$$

the single-particle energies are

$$\varepsilon_N = -V_0 + \left(N + \frac{3}{2}\right)\hbar\omega$$

$$N = n_x + n_y + n_z = 2n_r + l.$$

The zero-point energy per nucleon, measured upward from the potential minimum, is

$$E_{zp} = \frac{3}{2}\hbar\omega,$$

or approximately $61.5A^{-1/3}$ MeV when the phenomenological estimate $\hbar\omega \simeq 41A^{-1/3}$ MeV is used.

Including spin, the N -th oscillator shell can contain

$$g_N = (N + 1)(N + 2)$$

protons or neutrons separately, giving the predicted closures

$$2, 8, 20, 40, 70, 112, \dots$$

Thus the oscillator approximation is useful near the centre of a spherical nuclear mean field and explains the first three magic numbers, but it fails for higher

shells because it neglects the finite-surface form of the potential and, most importantly, the strong spin-orbit splitting. The lowering of the aligned $j = l + \frac{1}{2}$ orbitals reorganizes the spectrum and produces the observed higher magic numbers

$$28, 50, 82, 126.$$

CSE · 2026 · 10 MARKS

Q5. What do you understand by the packing fraction ' f ' and the binding energy ' E_b ' of a nucleus? Draw the graphs for packing fraction f versus mass number (A) and binding energy fraction $f_b \left(= \frac{E_b}{A} \right)$ versus mass number (A). Further, explain how the graphs of the variation of f and f_b with A have complementary approaches.

INTRODUCTION:

The question requires the definitions of packing fraction and nuclear binding energy, their qualitative variations with mass number A , and the physical reason for the complementary nature of these variations.

Consider a nuclide containing Z protons and $N = A - Z$ neutrons. Let $M(A, Z)$ be its ground-state atomic mass, u the unified atomic mass unit, m_H the mass of a hydrogen atom, and m_n the neutron mass. The binding-energy sign convention adopted here is

$$E_b > 0$$

for a bound nucleus; thus E_b is the energy required to separate the nucleus completely into free nucleons at rest and at infinite separation. Atomic masses will be used, with electronic binding-energy corrections neglected because they are very small on the nuclear-energy scale.

Since different isobars with the same A have different masses, the smooth curves represent nuclei on or near the valley of stability. Shell, pairing, and odd-even effects produce local deviations from these average trends.

SOLUTION:

1. The packing fraction is the fractional excess or deficiency of the actual isotopic mass over the integral mass number, conventionally multiplied by 10^4 :

$$f = 10^4 \frac{M/u - A}{A} = 10^4 \frac{M - Au}{Au}.$$

It is dimensionless. Some authors omit the factor 10^4 ; that convention changes only the numerical scale of the vertical axis.

The sign convention is therefore

$$\begin{aligned} f &> 0 \\ \iff \\ M &> Au \\ f &< 0 \\ \iff \\ M &< Au. \end{aligned}$$

Thus a negative packing fraction means that the average atomic mass per nucleon is less than u .

2. The nuclear binding energy is the work required to disintegrate the nucleus into its Z free protons and N free neutrons. It is also the energy released when these free nucleons combine to form the nucleus. In terms of nuclear masses,

$$E_b = [Zm_p + Nm_n - m_{\text{nuc}}(A, Z)] c^2.$$

Using atomic masses and neglecting the small electronic binding-energy correction,

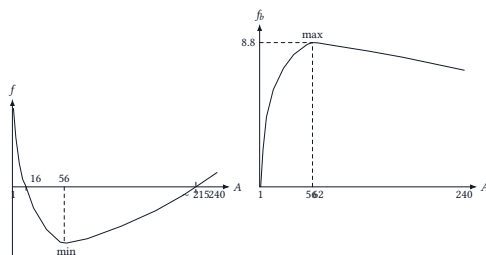
$$E_b \simeq [Zm_H + (A - Z)m_n - M(A, Z)] c^2.$$

The binding energy per nucleon, called the binding-energy fraction in the question, is

$$f_b = \frac{E_b}{A},$$

usually expressed in MeV nucleon^{-1} . It measures the average energy needed to remove a nucleon, although individual separation energies need not equal E_b/A .

3. The required qualitative graphs are shown below. The first zero crossing of the smooth packing-fraction curve occurs near $A \simeq 15$ –16; the light nuclei with $A \simeq 12$ –15 are therefore not to be represented on the negative branch of the smooth curve.



For the lightest nuclei, f is positive and falls rapidly with increasing A . Its smooth trend crosses the zero line near $A \simeq 15$ –16, becomes negative, and reaches a broad minimum

in the iron–nickel region. It then rises gradually and becomes positive again for sufficiently heavy nuclei, with the second crossing occurring roughly in the $A \simeq 210$ –220 region.

For a single free nucleon, $E_b = 0$, so $f_b = 0$. As nucleons form light nuclei, f_b rises rapidly. It attains a broad maximum of approximately

$$f_b \simeq 8.7\text{--}8.8 \text{ MeV nucleon}^{-1}$$

in the iron–nickel region, approximately $A \simeq 56$ –62, and then decreases slowly for heavier nuclei.

4. The complementary behavior follows directly from mass–energy conservation. The atomic-mass expression for binding energy gives

$$M(A, Z) \simeq Zm_H + Nm_n - \frac{E_b}{c^2}.$$

Subtracting Au , where $A = Z + N$, yields

$$M - Au \simeq Z(m_H - u) + N(m_n - u) - \frac{E_b}{c^2}.$$

Dividing by Au and using $f_b = E_b/A$,

$$\frac{M - Au}{Au} \simeq \frac{Z}{A} \left(\frac{m_H}{u} - 1 \right) + \frac{N}{A} \left(\frac{m_n}{u} - 1 \right) - \frac{f_b}{uc^2}.$$

Hence

$$\frac{f}{10^4} \simeq \frac{Z}{A} \left(\frac{m_H}{u} - 1 \right) + \frac{N}{A} \left(\frac{m_n}{u} - 1 \right) - \frac{f_b}{uc^2}.$$

The first two terms specify the mass per nucleon of the corresponding unbound proton–neutron mixture. Along the valley of stability they vary comparatively slowly with A , whereas the binding contribution enters with a negative sign. Greater binding energy per nucleon removes more rest mass per nucleon; consequently,

$$f_b \text{ increasing} \implies f \text{ decreasing},$$

and

$$f_b \text{ decreasing} \implies f \text{ increasing}.$$

Thus the broad maximum of f_b lies approximately in the same iron–nickel region in which f has its broad minimum.

The two graphs are complementary but are not exact mirror images. The reason is that the neutron-to-proton ratio, and therefore Z/A and N/A , changes systematically along the valley of stability. Shell structure, pairing, and odd–even effects also introduce local irregularities.

The maximum of f_b has an immediate energetic implication. Fusion of light nuclei can release energy when the product lies nearer the iron–nickel maximum and therefore has a larger E_b/A . Fission of a very heavy nucleus can likewise release energy when its medium-mass fragments have a larger E_b/A than the parent.

CONCLUSION:

For a nuclide of atomic mass $M(A, Z)$, the conventional packing fraction is

$$f = 10^4 \frac{M - Au}{Au},$$

and the positive binding energy and binding energy per nucleon are

$$E_b \approx [Zm_H + (A - Z)m_n - M(A, Z)]c^2$$

$$f_b = \frac{E_b}{A}.$$

Here f is dimensionless, E_b is an energy, and f_b is normally quoted in MeV nucleon⁻¹.

For nuclei near the valley of stability, the smooth $f(A)$ curve falls through zero near $A \approx 15$ – 16 , reaches a broad minimum near the iron–nickel region, and then rises, becoming positive again among very heavy nuclei. In the complementary variation, f_b rises from zero at $A = 1$ to approximately 8.7–8.8 MeV nucleon⁻¹ near $A \approx 56$ – 62 , and thereafter decreases slowly.

This complementarity results from the negative binding term in

$$\frac{f}{10^4} \approx \frac{Z}{A} \left(\frac{m_H}{u} - 1 \right) + \frac{N}{A} \left(\frac{m_n}{u} - 1 \right) - \frac{f_b}{uc^2}.$$

Subject to the gradual change of nuclear composition and local structure effects, a maximum of binding energy per nucleon therefore corresponds to a minimum of packing fraction. The iron–nickel region contains the most strongly bound nuclei on average, and motion toward this region permits energy release by fusion of light nuclei or fission of very heavy nuclei.

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Q6. What is the criticality of a self-sustained nuclear reactor? Write the basic processes affecting the nuclear chain reactions of a finite-size nuclear reactor. Also, write the critical size of reactors of different

shapes in terms of geometrical buckling constant.

INTRODUCTION:

A reactor is self-sustained when the neutron population reproduces itself from one generation to the next. Its criticality is therefore specified by the effective neutron multiplication factor

$$k_{\text{eff}} = \frac{\text{number of neutrons in one generation}}{\text{number of neutrons in the preceding generation}}.$$

Thus,

$$k_{\text{eff}} < 1$$

subcritical

$$k_{\text{eff}} = 1$$

critical

$$k_{\text{eff}} > 1$$

supercritical.

Equivalently, the reactivity is

$$\rho = \frac{k_{\text{eff}} - 1}{k_{\text{eff}}},$$

so that $\rho = 0$ at criticality.

For the size calculation, consider a bare, homogeneous reactor in the one-group diffusion approximation. Let $\phi(\mathbf{r})$ be the neutron flux, D the diffusion coefficient, Σ_a the macroscopic absorption cross-section, Σ_f the macroscopic fission cross-section, and ν the mean number of neutrons emitted per fission. The geometrical dimensions appearing in the diffusion boundary condition are extrapolated dimensions: the flux is taken to vanish a small extrapolation distance δ outside the physical surface.

SOLUTION:

The neutron life cycle in a finite reactor contains the following basic processes.

1. A neutron is absorbed in a fissile nucleus and produces fission. If ν neutrons are emitted per fission, this is the principal neutron-production process.
2. Some fast neutrons produce additional fissions before slowing down. This increases the neutron population by the fast-fission factor ϵ .
3. Elastic and inelastic scattering reduce neutron energies. In a thermal reactor, moderation carries the neutrons from fission energies to thermal energies, where the fission probability of fuels such as ^{235}U is large.
4. During slowing down, some neutrons undergo non-fission capture, particularly in resonance-energy regions. The resonance escape probability p is the fraction that reaches thermal energy without such absorption.

5. Thermal neutrons may be absorbed in the fuel, moderator, coolant, structural material, control material, or fission products. The thermal utilization factor f is the fraction of all thermal-neutron absorptions that occur in the fuel.
6. Absorption in the fuel may cause either fission or radiative capture. The reproduction factor η is the number of fission neutrons produced per thermal neutron absorbed in the fuel.
7. Since the reactor has finite dimensions, fast and thermal neutrons may escape through its boundary. If P_f and P_t are the fast and thermal non-leakage probabilities, respectively, leakage reduces the multiplication factor.

For a thermal reactor, these processes are summarized by the six-factor formula

$$k_{\text{eff}} = \eta \epsilon p f P_f P_t.$$

The first four factors give the infinite-medium multiplication factor,

$$k_{\infty} = \eta \epsilon p f,$$

whereas $P_f P_t < 1$ accounts for leakage from a finite reactor. A self-sustained steady chain reaction requires

$$\eta \epsilon p f P_f P_t = 1.$$

Delayed neutrons do not appreciably alter the static critical-size condition, but they are essential for the controllable time-dependent response of a reactor near criticality.

In one-group diffusion theory, the neutron balance eigenvalue equation is

$$-D\nabla^2\phi + \Sigma_a\phi = \frac{\nu\Sigma_f}{k_{\text{eff}}}\phi.$$

At exact criticality, $k_{\text{eff}} = 1$, and hence

$$-D\nabla^2\phi + \Sigma_a\phi = \nu\Sigma_f\phi.$$

Therefore,

$$\nabla^2\phi + B_m^2\phi = 0, \quad B_m^2 = \frac{\nu\Sigma_f - \Sigma_a}{D},$$

where B_m^2 is the material buckling. It measures the excess neutron production available to compensate leakage.

For a reactor of a specified shape, the fundamental spatial mode satisfies

$$\nabla^2\phi + B_g^2\phi = 0,$$

with zero flux at the extrapolated boundary. Here B_g^2 is the geometrical buckling, which measures leakage

imposed by the size and shape. The criticality condition is consequently

$$B_g^2 = B_m^2.$$

For a fixed material, a smaller reactor has larger B_g^2 and hence excessive leakage, whereas a larger reactor has smaller B_g^2 .

For the common reactor geometries, the critical dimensions follow as below.

1. For a slab of physical thickness T , infinite in the other two directions, let

$$T_e = T + 2\delta$$

be its extrapolated thickness. The fundamental flux is proportional to

$$\sin\left(\frac{\pi x}{T_e}\right),$$

and therefore

$$B_g^2 = \left(\frac{\pi}{T_e}\right)^2.$$

At criticality,

$$T_e = \frac{\pi}{B_m}, \quad T_c = \frac{\pi}{B_m} - 2\delta.$$

2. For a rectangular parallelepiped having physical side lengths a , b , and c , define

$$a_e = a + 2\delta, \quad b_e = b + 2\delta, \quad c_e = c + 2\delta.$$

Separation of variables gives the fundamental mode

$$\phi \propto \sin\left(\frac{\pi x}{a_e}\right) \sin\left(\frac{\pi y}{b_e}\right) \sin\left(\frac{\pi z}{c_e}\right),$$

so that

$$B_g^2 = \pi^2 \left(\frac{1}{a_e^2} + \frac{1}{b_e^2} + \frac{1}{c_e^2} \right).$$

The critical dimensions satisfy

$$B_m^2 = \pi^2 \left[\frac{1}{(a + 2\delta)^2} + \frac{1}{(b + 2\delta)^2} + \frac{1}{(c + 2\delta)^2} \right].$$

For a cube of physical side L ,

$$B_g^2 = \frac{3\pi^2}{(L + 2\delta)^2}, \quad L_c = \frac{\sqrt{3}\pi}{B_m} - 2\delta.$$

3. For a finite right circular cylinder of physical radius R and height H , take

$$R_e = R + \delta, \quad H_e = H + 2\delta.$$

The fundamental radial solution is $J_0(2.405 r/R_e)$, since 2.405 is the first positive zero of J_0 . Including the axial mode gives

$$B_g^2 = \left(\frac{2.405}{R_e}\right)^2 + \left(\frac{\pi}{H_e}\right)^2.$$

Thus its critical dimensions obey

$$B_m^2 = \left(\frac{2.405}{R + \delta}\right)^2 + \left(\frac{\pi}{H + 2\delta}\right)^2.$$

A unique pair (R_c, H_c) is obtained only after specifying the cylinder's aspect ratio or one of its dimensions.

4. For a sphere of physical radius R , let

$$R_e = R + \delta.$$

The finite spherically symmetric solution is

$$\phi(r) = C \frac{\sin(B_g r)}{r}.$$

Regularity at $r = 0$ selects this solution, and the extrapolated boundary condition $\phi(R_e) = 0$ gives

$$B_g R_e = \pi.$$

Hence

$$B_g^2 = \left(\frac{\pi}{R_e}\right)^2, \quad R_e = \frac{\pi}{B_m},$$

and the physical critical radius is

$$R_c = \frac{\pi}{B_m} - \delta.$$

Since B_m and B_g both have the dimension of inverse length, their equality consistently determines a critical length. The relations also show the expected physical limit: increasing a reactor dimension decreases B_g^2 , reduces neutron leakage, and moves the reactor toward supercriticality; decreasing the dimension increases leakage and moves it toward subcriticality.

CONCLUSION:

A self-sustained nuclear reactor is critical when

$$k_{\text{eff}} = 1, \quad \rho = 0,$$

so that neutron production exactly balances absorption and outward leakage. In life-cycle notation this is

$$\eta \epsilon p f P_f P_t = 1.$$

In the bare, homogeneous, one-group diffusion model, the equivalent critical-size condition is

$$B_g^2 = B_m^2 = \frac{\nu \Sigma_f - \Sigma_a}{D},$$

where B_g and B_m have units of inverse length.

Using extrapolated dimensions, the geometrical bucklings are

$$B_g^2 = \begin{cases} \left(\frac{\pi}{T_e}\right)^2, & \text{slab,} \\ \pi^2 \left(\frac{1}{a_e^2} + \frac{1}{b_e^2} + \frac{1}{c_e^2}\right), & \text{rectangular parallelepiped,} \\ \left(\frac{2.405}{R_e}\right)^2 + \left(\frac{\pi}{H_e}\right)^2, & \text{finite cylinder,} \\ \left(\frac{\pi}{R_e}\right)^2, & \text{sphere.} \end{cases}$$

Accordingly,

$$T_c = \frac{\pi}{B_m} - 2\delta$$

$$L_c = \frac{\sqrt{3}\pi}{B_m} - 2\delta$$

$$R_{c,\text{sphere}} = \frac{\pi}{B_m} - \delta$$

while the critical radius and height of a finite cylinder satisfy

$$\left(\frac{2.405}{R_c + \delta}\right)^2 + \left(\frac{\pi}{H_c + 2\delta}\right)^2 = B_m^2.$$

These results apply within diffusion theory to a bare homogeneous reactor; reflectors, heterogeneous cores, multigroup effects, and anisotropic leakage require the corresponding modified boundary conditions and neutron-balance equations.

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Q7. The total binding energies of ${}^{15}_8\text{O}$, ${}^{16}_8\text{O}$ and ${}^{17}_8\text{O}$ are 111.96 MeV, 127.62 MeV and 131.76 MeV respectively. Determine the energy gap between $1p_{1/2}$ and $1d_{5/2}$ neutron shells for the nuclide whose mass number is close to 16.

INTRODUCTION:

The proton number is fixed at $Z = 8$, while the neutron numbers of the three isotopes are

$$N({}^{15}\text{O}) = 7, \quad N({}^{16}\text{O}) = 8, \quad N({}^{17}\text{O}) = 9.$$

In the independent-particle shell model, ${}^{16}\text{O}$ is a doubly magic closed-shell nucleus. Its highest occupied

neutron orbit is $1p_{1/2}$, whereas the next neutron added to form ^{17}O occupies the $1d_{5/2}$ orbit.

Let $B(A)$ denote the positive total binding energy of ^AO . The one-neutron separation energy is

$$S_n(A) = B(A) - B(A - 1).$$

Neglecting changes in the mean nuclear field and residual-particle correlations among the three neighbouring isotopes, $S_n(^{16}\text{O})$ gives the binding of a neutron in the last occupied $1p_{1/2}$ orbit, while $S_n(^{17}\text{O})$ gives the binding of the added neutron in the $1d_{5/2}$ orbit.

Single-particle energies are measured relative to the neutron-emission threshold, so bound-state energies are negative:

$$\epsilon_{ntj} \approx -S_n.$$

The required shell gap is therefore

$$\Delta E = \epsilon(1d_{5/2}) - \epsilon(1p_{1/2}).$$

SOLUTION:

The energy required to remove the eighth neutron from ^{16}O , leaving ^{15}O , is

$$\begin{aligned} S_n(^{16}\text{O}) &= B(^{16}\text{O}) - B(^{15}\text{O}) \\ &= 127.62 \text{ MeV} - 111.96 \text{ MeV} \\ &= 15.66 \text{ MeV}. \end{aligned}$$

Thus, in the shell-model approximation,

$$\epsilon(1p_{1/2}) \approx -15.66 \text{ MeV}.$$

Similarly, the energy required to remove the ninth neutron from ^{17}O , leaving the closed-shell ^{16}O core, is

$$\begin{aligned} S_n(^{17}\text{O}) &= B(^{17}\text{O}) - B(^{16}\text{O}) \\ &= 131.76 \text{ MeV} - 127.62 \text{ MeV} \\ &= 4.14 \text{ MeV}. \end{aligned}$$

Hence,

$$\epsilon(1d_{5/2}) \approx -4.14 \text{ MeV}.$$

The separation between the two single-neutron levels is consequently

$$\begin{aligned} \Delta E &= \epsilon(1d_{5/2}) - \epsilon(1p_{1/2}) \\ &= (-4.14) - (-15.66) \text{ MeV} \\ &= 11.52 \text{ MeV}. \end{aligned}$$

Equivalently,

$$\Delta E = S_n(^{16}\text{O}) - S_n(^{17}\text{O}) = 2B(^{16}\text{O}) - B(^{15}\text{O}) - B(^{17}\text{O}).$$

The positive value is physically consistent: the occupied $1p_{1/2}$ neutron is more deeply bound than the $1d_{5/2}$ neutron lying above the $N = 8$ shell closure.

CONCLUSION:

The neutron-shell energy gap near mass number $A = 16$ is

$$\Delta E = \epsilon(1d_{5/2}) - \epsilon(1p_{1/2}) = 11.52 \text{ MeV}.$$

Thus the $1d_{5/2}$ neutron level lies approximately 11.52 MeV above the $1p_{1/2}$ level. This inference assumes that the neighbouring oxygen isotopes can be described as one-neutron particle or hole states about the closed-shell ^{16}O core, with changes of the mean field and residual correlation energies neglected.

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Q8. What is the minimum energy required to break a ^4_2He nucleus into free protons and neutrons? [Given, $m_{\text{H}} = 1.007825 \text{ amu}$, $m_n = 1.008665 \text{ amu}$, $m_e = 0.00055 \text{ amu}$ and $m_{\text{He}} = 4.002603 \text{ amu}$]

INTRODUCTION:

The ^4_2He nucleus contains two protons and two neutrons. The minimum energy required to separate it into free nucleons is its nuclear binding energy. It is obtained from the mass defect,

$$\Delta m = (2m_p + 2m_n) - m_{^4\text{He nucleus}},$$

using Einstein's mass-energy relation,

$$E_{\text{min}} = \Delta m c^2.$$

The tabulated masses m_{H} and m_{He} are atomic masses. Hence

$$m_p = m_{\text{H}} - m_e, \quad m_{^4\text{He nucleus}} = m_{\text{He}} - 2m_e,$$

where the negligible atomic electron binding energies are ignored.

SOLUTION:

Substituting the nuclear masses into the mass-defect expression gives

$$\begin{aligned} \Delta m &= 2(m_{\text{H}} - m_e) + 2m_n - (m_{\text{He}} - 2m_e) \\ &= 2m_{\text{H}} + 2m_n - m_{\text{He}}. \end{aligned}$$

Thus the electron masses cancel, as expected because both sides contain two electrons when atomic masses are used consistently.

Numerically,

$$\begin{aligned} \Delta m &= [2(1.007825) + 2(1.008665) - 4.002603] \text{ amu} \\ &= [2.015650 + 2.017330 - 4.002603] \text{ amu} \\ &= 0.030377 \text{ amu}. \end{aligned}$$

Using

$$1 \text{ amu } c^2 \approx 931.5 \text{ MeV},$$

the required energy is

$$E_{\min} = (0.030377 \text{ amu})(931.5 \text{ MeV/amu}) \\ \approx 28.30 \text{ MeV}.$$

Equivalently, since $1 \text{ MeV} = 1.602 \times 10^{-13} \text{ J}$,

$$E_{\min} = (28.30)(1.602 \times 10^{-13}) \text{ J} \\ \approx 4.53 \times 10^{-12} \text{ J}.$$

The positive mass defect confirms that energy must be supplied to overcome the binding of the nucleons. This value also corresponds to an average binding energy of

$$\frac{28.30}{4} \approx 7.07 \text{ MeV}$$

per nucleon, consistent with the strong binding of the ${}^4\text{He}$ nucleus.

CONCLUSION:

The minimum energy required to break one ${}^4_2\text{He}$ nucleus completely into two free protons and two free neutrons is

$$E_{\min} \approx 28.3 \text{ MeV} \approx 4.53 \times 10^{-12} \text{ J}.$$

This is the total binding energy of the ${}^4_2\text{He}$ nucleus and is the threshold energy, neglecting recoil and atomic electron-binding corrections.

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Q9. Consider a uranium nucleus (${}_{92}\text{U}^{236}$) breaking up spontaneously into two equal parts. Estimate the reduction of electrostatic energy of the nucleus considering uniform charge distribution. [Assume that nuclear radius is $1.2 \times 10^{-13} A^{1/3} \text{ cm}$]

INTRODUCTION:

The electrostatic self-energy of a uniformly charged spherical nucleus of charge $Q = Ze$ and radius R is

$$E_C = \frac{3}{5} \frac{1}{4\pi\epsilon_0} \frac{Q^2}{R}.$$

We compare the self-energy of the original spherical nucleus with the sum of the self-energies of two equal spherical fragments after they are sufficiently separated that their mutual electrostatic interaction is negligible.

For the parent nucleus,

$$A = 236, \quad Z = 92, \quad Q = 92e.$$

Equal division, with the proton-to-nucleon ratio unchanged, gives

$$A_f = \frac{236}{2} = 118, \quad Z_f = \frac{92}{2} = 46,$$

so charge and nucleon number are conserved. Writing the parent and fragment radii as R and R_f , respectively, the given radius law implies

$$R = r_0 A^{1/3}, \quad R_f = r_0 \left(\frac{A}{2}\right)^{1/3} = \frac{R}{2^{1/3}},$$

where

$$r_0 = 1.2 \times 10^{-13} \text{ cm} = 1.2 \text{ fm}.$$

SOLUTION:

The radius of the original uranium nucleus is

$$R = (1.2 \text{ fm})(236)^{1/3}.$$

Since

$$236^{1/3} \approx 6.18,$$

we obtain

$$R \approx (1.2)(6.18) \text{ fm} \approx 7.42 \text{ fm}.$$

Using

$$\frac{e^2}{4\pi\epsilon_0} = 1.44 \text{ MeV fm},$$

the electrostatic self-energy of the parent nucleus is

$$E_C^{(i)} = \frac{3}{5} \frac{1}{4\pi\epsilon_0} \frac{Z^2 e^2}{R}.$$

Therefore,

$$E_C^{(i)} = \frac{3}{5} \frac{(92)^2 (1.44 \text{ MeV fm})}{7.42 \text{ fm}}.$$

Numerically,

$$E_C^{(i)} = \frac{0.6 \times 8464 \times 1.44}{7.42} \text{ MeV} \approx 9.86 \times 10^2 \text{ MeV}.$$

Each fragment has charge $Ze/2$ and radius $R/2^{1/3}$. Its self-energy is

$$E_{C,f} = \frac{3}{5} \frac{1}{4\pi\epsilon_0} \frac{(Ze/2)^2}{R/2^{1/3}}.$$

The total self-energy of both fragments is consequently

$$E_C^{(f)} = 2E_{C,f} \\ = 2 \left[\frac{3}{5} \frac{1}{4\pi\epsilon_0} \frac{Z^2 e^2 / 4}{R/2^{1/3}} \right] \\ = \frac{3}{5} \frac{1}{4\pi\epsilon_0} \frac{Z^2 e^2}{R} 2^{-2/3} \\ = E_C^{(i)} 2^{-2/3}.$$

Since

$$2^{-2/3} \approx 0.630,$$

the final electrostatic self-energy is

$$E_C^{(f)} \approx (9.86 \times 10^2)(0.630) \text{ MeV} \approx 6.21 \times 10^2 \text{ MeV}.$$

Hence the reduction in electrostatic energy is

$$\begin{aligned} \Delta E_C &= E_C^{(i)} - E_C^{(f)} \\ &= E_C^{(i)} \left(1 - 2^{-2/3}\right) \\ &\approx (9.86 \times 10^2)(1 - 0.630) \text{ MeV} \\ &\approx 3.65 \times 10^2 \text{ MeV}. \end{aligned}$$

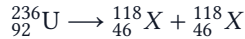
In SI units,

$$\Delta E_C \approx (3.65 \times 10^2)(1.602 \times 10^{-13}) \text{ J} \approx 5.85 \times 10^{-11} \text{ J}.$$

The positive value is physically consistent: division spreads the same total charge over two separated nuclei, thereby lowering the total electrostatic self-energy.

CONCLUSION:

For uniform spherical charge distributions, the spontaneous equal division



reduces the electrostatic self-energy by

$$\Delta E_C \approx 3.65 \times 10^2 \text{ MeV}$$

or equivalently

$$\Delta E_C \approx 5.85 \times 10^{-11} \text{ J}.$$

Thus approximately 365 MeV of electrostatic energy is released in the direction of fragment separation, under the assumptions of equal charge division, spherical uniform-charge fragments, unchanged nuclear density, and negligible mutual Coulomb interaction in the final sufficiently separated state.

CSE • 2025 • 10+10=20 MARKS

Q10. State the basic assumption of single-particle shell model. How do the centrifugal and spin-orbit terms remove the degeneracy of three-dimensional spherical harmonic oscillator?

INTRODUCTION:

The single-particle shell model assumes that, to a good approximation, each nucleon moves independently in a time-independent average potential produced by all the other nucleons. The complicated

residual two-nucleon interactions are replaced at leading order by a common, spherically symmetric mean field. The Pauli exclusion principle is imposed separately on protons and neutrons, and the residual interaction is treated as a comparatively small correction.

For a nucleon of mass m , the single-particle Hamiltonian may be written as

$$H = \frac{\mathbf{p}^2}{2m} + V(r) + V_{ls}(r) \mathbf{l} \cdot \mathbf{s}.$$

For the ideal three-dimensional isotropic harmonic oscillator,

$$V(r) = \frac{1}{2} m \omega^2 r^2$$

$$E_N^{(0)} = \left(N + \frac{3}{2}\right) \hbar \omega$$

$$N = 2n_r + l$$

where $n_r = 0, 1, 2, \dots$ is the radial quantum number and l is the orbital angular-momentum quantum number. Thus all states having the same oscillator quantum number N , but different allowed values

$$l = N, N-2, \dots,$$

have the same energy. Including spin, a major oscillator shell has degeneracy

$$g_N = 2 \sum_l (2l+1) = (N+1)(N+2).$$

The centrifugal contribution is examined in the radial equation, while the spin-orbit interaction is evaluated in the coupled basis

$$\mathbf{j} = \mathbf{l} + \mathbf{s}, \quad s = \frac{1}{2}.$$

SOLUTION:

For a central potential, write the wavefunction as

$$\psi_{nljm}(\mathbf{r}, \sigma) = \frac{u_{nl}(r)}{r} \mathcal{Y}_{ljm}(\hat{\mathbf{r}}, \sigma).$$

The radial Schrödinger equation is

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + V(r) + \frac{\hbar^2 l(l+1)}{2mr^2} \right] u_{nl}(r) = E_{nl} u_{nl}(r).$$

Hence the effective radial potential is

$$V_{\text{eff}}(r) = V(r) + V_{\text{cf}}(r), \quad V_{\text{cf}}(r) = \frac{\hbar^2 l(l+1)}{2mr^2}.$$

The centrifugal barrier vanishes for $l = 0$ and increases with $l(l+1)$. Consequently, states with different l experience different effective radial potentials and have different radial distributions. In a general

nuclear mean field this makes the energy depend separately on n_r and l :

$$E_{n_r l} \neq E_{n_r l'} \quad \text{for } l \neq l',$$

even when $2n_r + l = 2n_r' + l'$. The accidental degeneracy between different l -values within one oscillator shell is therefore removed.

There is an important qualification. The centrifugal term is already present in the exact radial solution of the isotropic harmonic oscillator. For the special potential $V(r) = m\omega^2 r^2/2$, its l -dependence is exactly compensated by the radial excitation energy, giving

$$E_{n_r l}^{(0)} = \left(2n_r + l + \frac{3}{2}\right)\hbar\omega = \left(N + \frac{3}{2}\right)\hbar\omega.$$

Thus the centrifugal barrier by itself does not split an *exact* isotropic oscillator shell. The l -splitting arises when the oscillator is replaced by, or corrected toward, a realistic finite-depth nuclear mean field. It is often represented phenomenologically within a major shell by an l -dependent term such as

$$\Delta H_l = D l^2,$$

where the coefficient D , including its sign, depends on the chosen mean-field parametrization. Its eigenvalue is

$$\Delta E_l = D \hbar^2 l(l+1),$$

so states with different l are no longer degenerate. Equivalently, the centrifugal barrier in the radial equation supplies the physical origin of the sensitivity to l , while the non-oscillator form of the central potential prevents the exact oscillator cancellation.

The remaining degeneracy between the two possible orientations of spin relative to the orbital angular momentum is removed by the spin-orbit interaction. Let

$$H_{ls} = V_{ls}(r) \mathbf{l} \cdot \mathbf{s}.$$

Using

$$\mathbf{j}^2 = (\mathbf{l} + \mathbf{s})^2 = \mathbf{l}^2 + \mathbf{s}^2 + 2\mathbf{l} \cdot \mathbf{s},$$

one obtains

$$\mathbf{l} \cdot \mathbf{s} = \frac{1}{2} (\mathbf{j}^2 - \mathbf{l}^2 - \mathbf{s}^2).$$

Therefore, in a state labelled by l, s, j ,

$$\langle \mathbf{l} \cdot \mathbf{s} \rangle = \frac{\hbar^2}{2} [j(j+1) - l(l+1) - s(s+1)].$$

For a nucleon, $s = \frac{1}{2}$, so $s(s+1) = 3/4$, and the allowed total angular momenta are

$$j = l + \frac{1}{2} \quad \text{and} \quad j = l - \frac{1}{2} \quad (l \neq 0).$$

For $j = l + \frac{1}{2}$,

$$j(j+1) = \left(l + \frac{1}{2}\right)\left(l + \frac{3}{2}\right) = l(l+1) + l + \frac{3}{4},$$

and hence

$$\langle \mathbf{l} \cdot \mathbf{s} \rangle_{j=l+1/2} = \frac{\hbar^2 l}{2}.$$

For $j = l - \frac{1}{2}$,

$$j(j+1) = \left(l - \frac{1}{2}\right)\left(l + \frac{1}{2}\right) = l^2 - \frac{1}{4},$$

which gives

$$\langle \mathbf{l} \cdot \mathbf{s} \rangle_{j=l-1/2} = -\frac{\hbar^2(l+1)}{2}.$$

If the radial dependence is replaced by its expectation value in the orbit,

$$C_{nl} = \langle V_{ls}(r) \rangle_{nl},$$

the first-order shifts are

$$\Delta E_{ls} \left(l + \frac{1}{2}\right) = C_{nl} \frac{\hbar^2 l}{2},$$

and

$$\Delta E_{ls} \left(l - \frac{1}{2}\right) = -C_{nl} \frac{\hbar^2(l+1)}{2}.$$

The separation between the two spin-orbit partners is consequently

$$E_{l+1/2} - E_{l-1/2} = C_{nl} \frac{\hbar^2}{2} (2l+1).$$

Thus the magnitude of the splitting increases approximately with $2l+1$.

In nuclei the effective spin-orbit coupling has the sign for which the aligned state

$$j = l + \frac{1}{2}$$

lies below the anti-aligned state

$$j = l - \frac{1}{2}.$$

With the convention used above, this corresponds to $C_{nl} < 0$. Each l -multiplet, originally containing

$$2(2l+1)$$

states including spin, is divided into two levels:

$$j = l + \frac{1}{2}, \quad 2j+1 = 2l+2,$$

and

$$j = l - \frac{1}{2}, \quad 2j+1 = 2l.$$

Their total number of magnetic substates is conserved:

$$(2l+2) + (2l) = 2(2l+1).$$

For $l = 0$, only $j = 1/2$ exists and

$$\mathbf{l} \cdot \mathbf{s} = 0,$$

so there is no spin-orbit splitting of an s -orbit.

CONCLUSION:

The basic shell-model assumption is that every proton or neutron moves approximately independently in a common, spherically symmetric mean field generated by all the other nucleons, with the Pauli principle enforced and residual interactions treated as corrections.

The ideal three-dimensional harmonic oscillator has energies

$$E_N^{(0)} = \left(N + \frac{3}{2}\right) \hbar\omega, \quad N = 2n_r + l,$$

so different l -orbits belonging to the same N -shell are accidentally degenerate. The centrifugal barrier

$$\frac{\hbar^2 l(l+1)}{2mr^2}$$

makes different l -states experience different effective radial potentials. In an exact harmonic oscillator its

effect is cancelled in the total energy, but in a realistic non-oscillator nuclear mean field this cancellation is lost, producing an l -dependent splitting.

The spin-orbit interaction further divides every $l \neq 0$ orbit into the partners

$$j = l + \frac{1}{2} \quad \text{and} \quad j = l - \frac{1}{2},$$

with

$$\langle \mathbf{l} \cdot \mathbf{s} \rangle = \begin{cases} \frac{\hbar^2 l}{2}, & j = l + \frac{1}{2}, \\ -\frac{\hbar^2(l+1)}{2}, & j = l - \frac{1}{2}. \end{cases}$$

For the nuclear sign of the coupling, the aligned $j = l + \frac{1}{2}$ level is lower, and the splitting has magnitude

$$|E_{l+1/2} - E_{l-1/2}| = |C_n| \frac{\hbar^2}{2} (2l+1).$$

The combined l -dependent central-field effect and strong spin-orbit splitting destroy the large oscillator degeneracy and generate the observed nuclear single-particle shell structure.



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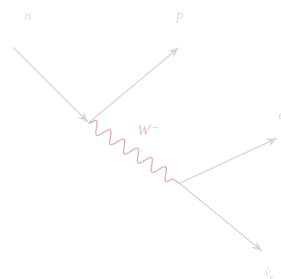
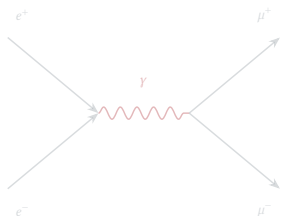
$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$E_n = -\frac{13.6}{n^2} \text{ eV}$$

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$$n \rightarrow p + e^- + \bar{\nu}_e$$

$$E^2 = p^2 c^2 + m^2 c^4$$

$$\partial_\mu j^\mu = 0$$

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Sample Preview extbar 10 Solved Questions

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CSE • 2026 • 5 MARKS

Q1. What are the different methods used to increase the probability of exposure for the atomic planes with right orientation to X-rays in the X-rays diffraction studies?

INTRODUCTION:

For diffraction from a set of parallel atomic planes of spacing d , the incident X-rays must satisfy Bragg's condition

$$2d \sin \theta = n\lambda,$$

where θ is the glancing angle between the incident beam and the atomic planes, λ is the X-ray wavelength, and n is the order of diffraction. Thus, for fixed d and λ , diffraction occurs only when the plane normal has the required orientation relative to the incident beam. The probability of meeting this condition can be increased by varying the wavelength, varying the crystal orientation, or simultaneously providing many randomly oriented crystallites.

SOLUTION:

The principal experimental methods are as follows.

1. Laue method: variation of wavelength

A stationary single crystal is exposed to a collimated beam of polychromatic, or "white," X-rays containing a continuous range of wavelengths. The orientations and spacings of the atomic planes remain fixed, but for a suitably oriented plane there is a greater probability that the incident spectrum contains a wavelength

$$\lambda = \frac{2d \sin \theta}{n}$$

which satisfies Bragg's law.

The diffraction spots may be recorded in transmission geometry, with the detector behind the crystal, or in back-reflection geometry, with the detector on the incident-beam side. This method is especially useful for determining crystal symmetry and orientation.

2. Rotating-crystal or oscillation method: variation of crystal orientation

A single crystal is irradiated with monochromatic X-rays and rotated continuously, or oscillated through a limited angular range, about a chosen axis. As the crystal turns, the angle θ made by each family of atomic planes with the incident beam changes. Different planes consequently pass through the required Bragg orientation

$$\theta = \sin^{-1} \left(\frac{n\lambda}{2d} \right).$$

Whenever a reciprocal-lattice point intersects the Ewald sphere, the corresponding reflected beam is recorded. Related procedures, such as the precession method, vary the crystal orientation in a controlled manner so that a larger and more conveniently arranged set of reflections is obtained.

3. Powder or Debye-Scherrer method: use of many random orientations

A finely powdered specimen contains a very large number of small crystallites whose orientations are approximately random. For any family of planes with spacing d , some crystallites will statistically have those planes at the Bragg angle θ to the incident monochromatic beam. Their diffracted rays have the same scattering angle 2θ and form a cone about the incident-beam direction. A film or position-sensitive detector intercepts this cone as rings or arcs.

The method therefore supplies the required orientations simultaneously without mechanically aligning a single crystal. It is particularly suitable for phase identification and determination of interplanar spacings.

These approaches correspond to the three quantities that can be diversified in Bragg's condition: λ is varied in the Laue method, θ is varied in rotating or oscillating-crystal methods, and a large statistical distribution of θ is supplied by the powder method. A reflection is possible only when $n\lambda \leq 2d$, since $\sin \theta \leq 1$.

CONCLUSION:

The probability of exposing atomic planes in the correct orientation is increased by:

1. using a stationary single crystal and many X-ray wavelengths in the Laue method;
2. rotating, oscillating, or precessing a single crystal in a monochromatic beam so that its plane normals successively acquire the required direction; and
3. using a powder containing numerous randomly oriented crystallites, some of which automatically have the required plane orientation.

In every case, the selected planes must be oriented at the Bragg glancing angle

$$\theta = \sin^{-1} \left(\frac{n\lambda}{2d} \right)$$

to the incident X-ray direction, equivalently with their normal making the angle $90^\circ - \theta$ with that direction.

CSE • 2026 • 15 MARKS

Q2. Explain the principle and working of the Laue's diffraction method. Comment on the origin of Laue spots and the utility of Laue's diffraction pattern.

INTRODUCTION:

Laue's method is an X-ray diffraction method in which a stationary single crystal is irradiated by a narrow beam of polychromatic, or "white," X-rays. Its principle is the coherent elastic scattering of X-rays by the periodically arranged atoms of the crystal. Although the orientation of the crystal is fixed, the continuous range of wavelengths allows different sets of lattice planes to satisfy the diffraction condition simultaneously.

Let the primitive translation vectors of the crystal be $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Let $\mathbf{k}_0$ and $\mathbf{k}$ denote the incident and diffracted wave vectors, respectively, with

$$|\mathbf{k}_0| = |\mathbf{k}| = \frac{2\pi}{\lambda},$$

because X-ray diffraction is treated as elastic scattering. The scattering vector is

$$\mathbf{q} = \mathbf{k} - \mathbf{k}_0.$$

Constructive interference occurs when $\mathbf{q}$ is a reciprocal-lattice vector. The incident-beam direction is taken as fixed, and the direction of each diffracted beam is determined by the corresponding reciprocal-lattice point and the wavelength selected from the incident spectrum.

SOLUTION:

Consider identical scatterers situated at lattice points

$$\mathbf{R} = u\mathbf{a} + v\mathbf{b} + w\mathbf{c}, \quad u, v, w \in \mathbb{Z}.$$

The phase difference between waves scattered from lattice points separated by $\mathbf{R}$ is

$$\Delta\phi = (\mathbf{k} - \mathbf{k}_0) \cdot \mathbf{R} = \mathbf{q} \cdot \mathbf{R}.$$

For constructive interference from the entire crystal, this phase difference must be an integral multiple of 2π for every lattice translation. It is therefore sufficient and necessary that

$$\mathbf{q} \cdot \mathbf{a} = 2\pi h, \quad \mathbf{q} \cdot \mathbf{b} = 2\pi k, \quad \mathbf{q} \cdot \mathbf{c} = 2\pi l,$$

where h, k, l are integers. These are the three Laue diffraction equations. Equivalently,

$$\boxed{\mathbf{k} - \mathbf{k}_0 = \mathbf{G}_{hkl}},$$

where

$$\mathbf{G}_{hkl} = h\mathbf{a}^* + k\mathbf{b}^* + l\mathbf{c}^*$$

is a reciprocal-lattice vector, with the reciprocal basis defined by

$$\mathbf{a}^* \cdot \mathbf{a} = 2\pi, \quad \mathbf{a}^* \cdot \mathbf{b} = \mathbf{a}^* \cdot \mathbf{c} = 0,$$

and corresponding cyclic relations for $\mathbf{b}^*$ and $\mathbf{c}^*$.

If $(\alpha_0, \beta_0, \gamma_0)$ are the angles made by the incident beam with $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and (α, β, γ) are those made by a diffracted beam, the Laue equations may also be written as

$$a(\cos \alpha - \cos \alpha_0) = h\lambda,$$

$$b(\cos \beta - \cos \beta_0) = k\lambda,$$

$$c(\cos \gamma - \cos \gamma_0) = l\lambda$$

for an orthogonal lattice with edge lengths a, b, c .

The connection with Bragg's law follows from elastic scattering. Using

$$\mathbf{k} = \mathbf{k}_0 + \mathbf{G}_{hkl}$$

and $|\mathbf{k}| = |\mathbf{k}_0|$, one obtains

$$|\mathbf{k}_0 + \mathbf{G}_{hkl}|^2 = |\mathbf{k}_0|^2,$$

so that

$$2\mathbf{k}_0 \cdot \mathbf{G}_{hkl} + |\mathbf{G}_{hkl}|^2 = 0.$$

Since $\mathbf{G}_{hkl}$ is normal to the corresponding family of lattice planes, this geometrical condition reduces to

$$\boxed{n\lambda = 2d_{hkl} \sin \theta},$$

where d_{hkl} is the interplanar spacing, θ is the glancing angle, and n is the diffraction order. Thus, the Laue equations and Bragg's law express the same condition in vector and plane descriptions, respectively.

The experimental working is as follows:

1. A beam of white X-rays, containing a continuous range of wavelengths, is produced by an X-ray tube and made narrow and approximately parallel by collimating slits.
2. A stationary single crystal is placed in the path of the collimated beam. Its orientation remains fixed throughout the exposure.
3. For a particular family of planes, d_{hkl} and θ are fixed because the crystal is stationary. From the incident continuous spectrum, only the wavelength

$$\lambda = \frac{2d_{hkl} \sin \theta}{n}$$

that lies within the available spectral range is strongly diffracted.

4. Many differently oriented plane families can select different wavelengths at the same time. Consequently, several diffracted beams emerge in different directions.

5. A photographic film or a two-dimensional X-ray detector records the intersections of these diffracted beams with the detector surface. The direct transmitted beam is normally intercepted or identified separately.

Two commonly used arrangements are:

1. In the transmission Laue method, the detector is placed behind the crystal, approximately normal to the incident beam. The forward-diffracted rays produce spots around the transmitted-beam position. This arrangement is suitable especially for relatively thin crystals.
2. In the back-reflection Laue method, the detector is placed on the incident-beam side of the crystal, with an opening through which the primary beam passes. The detector records rays diffracted through large angles. This arrangement is useful for thick or strongly absorbing specimens because the radiation need penetrate only a limited depth.

Each observed Laue spot originates from constructive interference associated with a reciprocal-lattice vector $\mathbf{G}_{hkl}$, or equivalently with a particular set of lattice planes. Since the crystal is not rotated, the spot does not arise because a plane family is mechanically brought into reflection position. Instead, that family selects from the white spectrum the wavelength that satisfies Bragg's law. Thus different spots generally correspond not only to different (hkl) plane families but also to different wavelengths.

In reciprocal-space language, the incident wave vectors of different magnitudes form a range of Ewald spheres because $|\mathbf{k}_0| = 2\pi/\lambda$ varies across the white spectrum. A reflection occurs whenever one of these spheres passes through a reciprocal-lattice point. The direction of the corresponding diffracted wave vector fixes the location of the spot on the detector.

The spots are discrete because the translational periodicity of a well-ordered crystal permits strong diffraction only at reciprocal-lattice points. Their intensities depend on the incident spectral intensity, polarization and geometrical factors, absorption, multiplicity, and especially the crystal structure factor

$$F_{hkl} = \sum_j f_j \exp[2\pi i (hx_j + ky_j + lz_j)],$$

where f_j and (x_j, y_j, z_j) are the atomic scattering factor and fractional coordinates of the j -th atom in the unit cell. A geometrically allowed spot may therefore be weak or absent when $F_{hkl} = 0$.

The Laue pattern has the following principal uses:

1. *Determination of crystal orientation:* The positions of the spots identify the directions of crys-

tallographic axes relative to the incident beam. The method is widely used to align single crystals before cutting, polishing, or carrying out other diffraction measurements.

2. *Identification of symmetry:* The symmetry of the spot pattern about the incident-beam direction reveals the corresponding rotational and mirror symmetry of the crystal orientation, subject to the diffraction symmetry and possible intensity equivalences.
3. *Testing whether a specimen is a single crystal:* A well-oriented single crystal produces an orderly array of sharp spots. Several differently oriented crystals or grains produce superposed sets of spots.
4. *Assessment of crystal quality:* Sharp, regular spots indicate good long-range order. Split, streaked, diffuse, broadened, or distorted spots can reveal mosaic structure, strain, lattice defects, bending, disorder, or imperfect crystallinity.
5. *Observation of orientation changes:* Changes in spot positions can be used to study crystal rotation, deformation, recrystallization, and related structural changes.
6. *Rapid crystallographic identification:* Comparison of the measured spot geometry with calculated or standard patterns helps identify crystallographic directions and, in suitable cases, distinguish possible crystal symmetries.

Because each spot normally involves a different and not always precisely known wavelength, an ordinary Laue photograph is less convenient for accurate determination of lattice constants or a complete crystal structure than monochromatic rotating-crystal or powder methods. Its chief strength is the rapid determination of orientation, symmetry, and crystal perfection.

CONCLUSION:

Laue's method uses a stationary single crystal and a collimated polychromatic X-ray beam. Diffraction occurs in directions satisfying

$$\mathbf{k} - \mathbf{k}_0 = \mathbf{G}_{hkl},$$

or, in direct-lattice form,

$$\mathbf{q} \cdot \mathbf{a} = 2\pi h, \quad \mathbf{q} \cdot \mathbf{b} = 2\pi k, \quad \mathbf{q} \cdot \mathbf{c} = 2\pi l.$$

For elastic scattering these conditions are equivalent to

$$n\lambda = 2d_{hkl} \sin \theta.$$

Each reciprocal-lattice point that can satisfy this condition selects an appropriate wavelength, measured in

units of length, from the white spectrum and sends a diffracted beam in a definite direction. Its intersection with the detector forms a Laue spot. The method is valid when the specimen has sufficient crystalline order, the scattering is predominantly elastic, and the required wavelength lies within the incident spectrum. The positions, symmetry, sharpness, splitting, and intensities of the spots provide rapid information about crystal orientation, symmetry, single-crystal character, strain, defects, and overall crystal quality.

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Q3. What is X-ray diffraction? How is an XRD pattern used to determine the crystal structure of the material?

INTRODUCTION:

X-ray diffraction (XRD) is the coherent elastic scattering of X-rays by the periodically arranged atoms of a crystalline material. Since X-ray wavelengths, typically of order 10^{-10} m, are comparable to interatomic spacings, waves scattered from different lattice planes interfere. Constructive interference produces diffraction peaks whose positions, intensities, and shapes contain information about the crystal structure.

Consider monochromatic radiation of wavelength λ incident on a set of parallel lattice planes separated by d_{hkl} , where (hkl) are Miller indices. Let θ be the glancing angle between the incident beam and the planes; the angle between the incident and diffracted beams is then 2θ . The discussion assumes a crystalline specimen, predominantly elastic and kinematic scattering, and an accurately known wavelength and instrumental geometry.

SOLUTION:

For rays scattered by two successive lattice planes, the extra distance travelled by the lower ray is

$$\Delta = 2d_{hkl} \sin \theta.$$

Constructive interference requires this path difference to be an integral multiple of the wavelength:

$$2d_{hkl} \sin \theta = n\lambda, \quad n = 1, 2, \dots$$

This is Bragg's law. Higher-order diffraction may equivalently be indexed as first-order diffraction from another reciprocal-lattice plane, so XRD patterns are ordinarily indexed using $n = 1$:

$$d_{hkl} = \frac{\lambda}{2 \sin \theta_{hkl}}.$$

An XRD pattern records diffracted intensity I as a function of 2θ . In a powder specimen, the crystal-

lites are randomly oriented; for every allowed family of lattice planes, some crystallites satisfy Bragg's condition. Consequently, each plane family produces a diffraction cone, detected as a peak at a definite value of 2θ .

The crystal structure is determined from the pattern as follows.

1. *Determine the interplanar spacings from peak positions.*

For every measured peak at $2\theta_i$, its Bragg angle is $\theta_i = (2\theta_i)/2$, and

$$d_i = \frac{\lambda}{2 \sin \theta_i}.$$

Thus the peak positions give a set of experimental interplanar spacings. These spacings are compared with the metric relations appropriate to possible crystal systems. For an orthogonal unit cell,

$$\frac{1}{d_{hkl}^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} + \frac{l^2}{c^2},$$

where a, b, c are the lattice parameters. Important special cases are

$$\begin{aligned} \text{cubic:} \quad & \frac{1}{d_{hkl}^2} = \frac{h^2 + k^2 + l^2}{a^2}, \\ \text{tetragonal:} \quad & \frac{1}{d_{hkl}^2} = \frac{h^2 + k^2}{a^2} + \frac{l^2}{c^2}, \\ \text{orthorhombic:} \quad & \frac{1}{d_{hkl}^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} + \frac{l^2}{c^2}. \end{aligned}$$

For a cubic crystal, Bragg's law gives

$$\sin^2 \theta_{hkl} = \frac{\lambda^2}{4a^2} (h^2 + k^2 + l^2).$$

Hence the ratios of experimental $\sin^2 \theta$ values must equal ratios of the allowed integers $h^2 + k^2 + l^2$. Matching these ratios assigns Miller indices to the peaks and determines the lattice constant through

$$a = d_{hkl} \sqrt{h^2 + k^2 + l^2} = \frac{\lambda \sqrt{h^2 + k^2 + l^2}}{2 \sin \theta_{hkl}}.$$

A consistent value of a from several independently indexed peaks is an important test of the proposed indexing.

2. *Identify the Bravais lattice from systematic absences.*

Not every geometrically possible reflection has nonzero intensity. Translational symmetry

causes characteristic reflection conditions. For conventional cubic lattices,

Bravais lattice	Allowed reflections
primitive cubic (P)	all integral h, k, l ,
body-centred cubic (I)	$h + k + l$ even,
face-centred cubic (F)	h, k, l all even or all odd.

These conditions follow from the lattice structure factor. For identical atoms with atomic scattering factor f ,

$$F_{hkl} = \sum_j f e^{2\pi i(hx_j + ky_j + lz_j)},$$

where (x_j, y_j, z_j) are fractional atomic coordinates.

For example, a body-centred lattice has atoms at $(0, 0, 0)$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, so

$$F_{hkl} = f \left[1 + e^{i\pi(h+k+l)} \right].$$

Therefore $F_{hkl} = 2f$ when $h + k + l$ is even, but $F_{hkl} = 0$ when it is odd. The missing odd-sum reflections consequently identify body centring. Analogous systematic absences distinguish face centring and other translational symmetries.

3. Determine the atomic basis from relative intensities.

The integrated intensity of a reflection is, in simplified form,

$$I_{hkl} \propto m_{hkl} L_{hkl} P_{hkl} |F_{hkl}|^2 e^{-2M_{hkl}},$$

where m_{hkl} is the multiplicity, $L_{hkl} P_{hkl}$ represents geometrical and polarization corrections, and $e^{-2M_{hkl}}$ is the thermal factor. For a basis containing different atoms,

$$F_{hkl} = \sum_j f_j e^{2\pi i(hx_j + ky_j + lz_j)}.$$

The factors f_j depend on the atomic species and scattering angle. Because the measured intensity is governed by $|F_{hkl}|^2$, different arrangements of atoms within the same unit cell produce different relative peak intensities even when their peak positions are identical. Trial structural models are therefore used to calculate intensities, which are compared with the observed values to infer the atomic species, fractional coordinates, site occupancies, and thermal displacements.

4. Refine and validate the structural model.

After determining a probable crystal system, Bravais lattice, unit-cell dimensions, and atomic basis, a calculated complete pattern is

fitted to the observed pattern. In Rietveld refinement, structural and profile parameters are varied simultaneously to minimize a weighted residual such as

$$S = \sum_i w_i \left[I_i^{\text{obs}} - I_i^{\text{calc}} \right]^2.$$

Agreement must extend over the entire pattern: peak positions test the unit-cell metric, systematic absences test lattice symmetry, relative intensities test the atomic basis, and peak profiles provide information about crystallite size, microstrain, defects, and instrumental broadening. Unindexed peaks may indicate an incorrect model or the presence of additional crystalline phases.

The diffraction condition also has a reciprocal-space interpretation. If $\mathbf{k}_0$ and $\mathbf{k}$ are the incident and elastically scattered wavevectors, with

$$|\mathbf{k}_0| = |\mathbf{k}| = \frac{2\pi}{\lambda},$$

then diffraction occurs when the scattering vector equals a reciprocal-lattice vector:

$$\mathbf{k} - \mathbf{k}_0 = \mathbf{G}_{hkl}.$$

This form makes clear that peak positions determine the reciprocal lattice, whereas intensities determine how atoms are distributed within the corresponding real-space unit cell.

XRD has intrinsic limitations. Since an ordinary diffraction measurement records $|F_{hkl}|^2$ rather than the phase of F_{hkl} , atomic positions cannot generally be obtained by direct inversion without additional structural constraints or phase-determination methods. Furthermore, amorphous materials produce broad diffuse features rather than a set of sharp Bragg peaks, and preferred orientation, overlapping peaks, multiphase samples, and light elements can complicate structure determination.

CONCLUSION:

X-ray diffraction is the coherent elastic scattering of X-rays by a periodic crystal lattice, with diffraction maxima satisfying

$$2d_{hkl} \sin \theta = n\lambda.$$

An XRD pattern determines crystal structure through complementary information: peak positions yield d_{hkl} , lattice parameters, and the crystal system; systematic absences identify translational symmetry and the Bravais lattice; relative integrated intensities, through

$$I_{hkl} \propto |F_{hkl}|^2,$$

determine the atomic basis; and whole-pattern refinement tests and improves the complete structural

model. The inferred structure is reliable when one indexed unit cell consistently accounts for the observed peak positions, extinction rules, and corrected intensities, subject to the assumptions of adequate crystallinity, known wavelength, suitable diffraction geometry, and proper allowance for instrumental and specimen effects.

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Q4. In powder diffraction method pattern for lead with radiation of wavelength $\lambda = 1.54 \text{ \AA}$, the (220) Bragg reflection angle is $\theta = 32^\circ$. Find the radius of the atom.

INTRODUCTION:

Lead crystallizes in the face-centred cubic (FCC) structure. We assume the conventional hard-sphere model in which neighbouring atoms touch along a face diagonal. The indexed (220) powder line is taken as a first-order Bragg reflection, and the given θ is the Bragg angle between the incident beam and the reflecting lattice planes.

Let a be the cubic lattice parameter, d_{220} the spacing of the (220) planes, and R the atomic radius. Bragg's law and the interplanar-spacing relation for a cubic crystal will be used to determine a , after which the FCC contact geometry gives R .

SOLUTION:

For first-order diffraction, Bragg's law is

$$\lambda = 2d_{220} \sin \theta.$$

Hence,

$$d_{220} = \frac{\lambda}{2 \sin \theta}.$$

For a cubic crystal,

$$d_{hkl} = \frac{a}{\sqrt{h^2 + k^2 + l^2}}.$$

Therefore, for the (220) planes,

$$d_{220} = \frac{a}{\sqrt{2^2 + 2^2 + 0^2}} = \frac{a}{\sqrt{8}} = \frac{a}{2\sqrt{2}}.$$

Substitution in Bragg's law gives

$$\lambda = 2 \left(\frac{a}{2\sqrt{2}} \right) \sin \theta = \frac{a \sin \theta}{\sqrt{2}},$$

so that

$$a = \frac{\sqrt{2}\lambda}{\sin \theta}.$$

In an FCC unit cell, atoms touch along a face diagonal. The face diagonal has length $a\sqrt{2}$ and contains four atomic radii:

$$4R = a\sqrt{2}.$$

Thus,

$$R = \frac{a\sqrt{2}}{4} = \frac{a}{2\sqrt{2}}.$$

Using the expression for a ,

$$R = \frac{1}{2\sqrt{2}} \left(\frac{\sqrt{2}\lambda}{\sin \theta} \right) = \frac{\lambda}{2 \sin \theta}.$$

Equivalently, for FCC lead, $R = d_{220}$.

Substituting the given values,

$$R = \frac{1.54 \text{ \AA}}{2 \sin 32^\circ} = \frac{1.54 \text{ \AA}}{2(0.5299)} = \frac{1.54 \text{ \AA}}{1.0598} \approx 1.45 \text{ \AA}.$$

In SI units,

$$R \approx 1.45 \times 10^{-10} \text{ m}.$$

The corresponding lattice parameter is

$$a = 2\sqrt{2} R \approx 2\sqrt{2}(1.45 \text{ \AA}) \approx 4.11 \text{ \AA},$$

which is physically consistent with an FCC metal having an atomic diameter $2R \approx 2.90 \text{ \AA}$.

CONCLUSION:

The radius of a lead atom, treating FCC atoms as touching hard spheres, is

$$R \approx 1.45 \text{ \AA} = 1.45 \times 10^{-10} \text{ m}.$$

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Q5. Find the radius of the interstitial sphere which can just fit into the void at the body centre of the fcc structure coordinated by the facial atoms.

INTRODUCTION:

Let the radius of each atom forming the ideal face-centred cubic structure be R , the cube-edge length be a , and the radius of the interstitial sphere be r . The atoms are treated as identical rigid spheres. In an fcc structure, neighbouring atoms touch along each face diagonal.

The void at the body centre is surrounded by the six face-centred atoms, one on each face of the cube, and is therefore an octahedral void of coordination number 6. Taking the body centre and a face centre on a common Cartesian axis, the distance between them is $a/2$. For a sphere that just fits into the void, it must touch each of these six atoms.

SOLUTION:

The length of a face diagonal of the cubic unit cell is

$$\sqrt{a^2 + a^2} = \sqrt{2}a.$$

Along this diagonal, the two corner atoms and the intervening face-centred atom are in contact. Hence the face diagonal equals four atomic radii:

$$\sqrt{2}a = 4R.$$

Therefore,

$$a = \frac{4R}{\sqrt{2}} = 2\sqrt{2}R.$$

The distance from the body centre,

$$\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right),$$

to the centre of the atom on, for example, the face $x = 0$,

$$\left(0, \frac{a}{2}, \frac{a}{2}\right),$$

is

$$d = \sqrt{\left(\frac{a}{2}\right)^2 + 0^2 + 0^2} = \frac{a}{2}.$$

When the interstitial sphere just fits, it touches the face-centred atom externally. Thus,

$$R + r = \frac{a}{2}.$$

Using $a = 2\sqrt{2}R$,

$$R + r = \frac{2\sqrt{2}R}{2} = \sqrt{2}R,$$

so that

$$r = (\sqrt{2} - 1)R.$$

Numerically,

$$\frac{r}{R} = \sqrt{2} - 1 = 1.4142 - 1 \approx 0.414.$$

The corner atoms are farther from the body centre than the facial atoms: their centre-to-centre distance is

$$\frac{\sqrt{3}a}{2} = \sqrt{6}R,$$

which exceeds $R + r = \sqrt{2}R$. Hence the six face-centred atoms, rather than the corner atoms, determine the limiting size of the interstitial sphere.

CONCLUSION:

The body-centred void in an ideal fcc structure is an octahedral void coordinated by six face-centred atoms. The radius of the largest interstitial sphere that can just fit into it is

$$r = (\sqrt{2} - 1)R \approx 0.414R,$$

where R is the radius of an atom of the fcc lattice. Thus, the interstitial sphere can have at most approximately 41.4% of the host-atom radius.

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Q6. (i) Explain the Bravais lattice and non-Bravais lattice with suitable diagram. (ii) Draw the (1 0 0), (1 1 0) and (1 1 1) planes in a cubic unit cell. (iii) Obtain the Miller indices of a plane with intercepts a , $b/2$, $3c$ in a simple cubic unit cell.

INTRODUCTION:

A Bravais lattice is an infinite periodic array of geometrically equivalent points generated by integral translations

$$\mathbf{R} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3, \quad n_1, n_2, n_3 \in \mathbb{Z},$$

where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are primitive translation vectors. Every Bravais-lattice point therefore has the same surroundings and can be carried into every other lattice point by a lattice translation.

The expression “non-Bravais lattice” is conventionally used for a periodic atomic arrangement whose sites cannot all be generated from one site by translations alone. More precisely, it is a crystal structure consisting of a Bravais lattice supplied with a basis of two or more inequivalent sites.

For the crystallographic parts, take a right-handed Cartesian frame with the origin at a corner of the unit cell and the positive x , y , and z axes along the crystallographic axes of lengths a , b , and c , respectively. A plane with intercepts pa , qb , and rc has Miller indices proportional to

$$\left(\frac{1}{p}, \frac{1}{q}, \frac{1}{r}\right),$$

after clearing fractions and reducing to the smallest relatively prime integers. An infinite intercept has reciprocal zero. For the cubic cell in part (ii), $a = b = c$, and the plane (hkl) may be represented by

$$h\frac{x}{a} + k\frac{y}{a} + l\frac{z}{a} = 1.$$

SOLUTION:

- (i) For a Bravais lattice, translation by any lattice vector $\mathbf{R}$ leaves the complete array invariant. Thus, if $\mathbf{R}_1$ and $\mathbf{R}_2$ are any two lattice points, then

$$\mathbf{R}_2 - \mathbf{R}_1$$

is a lattice translation, and the local environment at $\mathbf{R}_1$ is identical to that at $\mathbf{R}_2$ with the same orientation.

A square array containing one identical point per primitive cell is a two-dimensional example:

$$\mathbf{R} = n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2, \quad n_1, n_2 \in \mathbb{Z}.$$

In contrast, suppose every square Bravais-lattice point carries a two-site basis

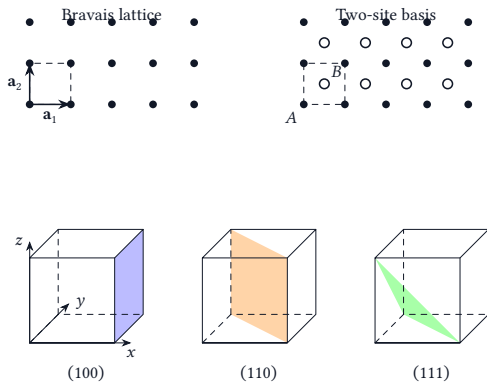
$$\boldsymbol{\tau}_A = \mathbf{0}, \quad \boldsymbol{\tau}_B = \frac{\mathbf{a}_1 + \mathbf{a}_2}{2},$$

where A and B denote inequivalent atoms or sites. The positions are then

$$\mathbf{r}_{n,s} = \mathbf{R}_n + \boldsymbol{\tau}_s, \quad s = A, B.$$

The translations $\mathbf{R}_n$ carry A sites into A sites and B sites into B sites, but no lattice translation interchanges inequivalent A and B sites. Hence the complete atomic arrangement is not itself a one-point Bravais lattice, although its translational skeleton is Bravais.

The upper panels of the following diagram illustrate these two cases. The lower panels show the requested planes in one cubic unit cell.



- (ii) In a cubic unit cell, the representative plane (hkl) cuts the axes at

$$x = \frac{a}{h}, \quad y = \frac{a}{k}, \quad z = \frac{a}{l},$$

with a zero index indicating that the plane is parallel to the corresponding axis.

For the three specified planes:

$$(100) : \quad \frac{x}{a} = 1 \implies x = a.$$

It is the face perpendicular to the positive x -axis and parallel to the yz -plane. Its intercepts are (a, ∞, ∞) .

$$(110) : \quad \frac{x}{a} + \frac{y}{a} = 1 \implies x + y = a.$$

It cuts the positive x - and y -axes at a and is parallel to the z -axis. Its intercepts are (a, a, ∞) .

$$(111) : \quad \frac{x}{a} + \frac{y}{a} + \frac{z}{a} = 1 \implies x + y + z = a.$$

It cuts all three positive axes at a , so the portion inside the unit cell is the triangle joining $(a, 0, 0)$, $(0, a, 0)$, and $(0, 0, a)$.

In a cubic crystal, the normal to a plane (hkl) is parallel to the crystallographic direction $[hkl]$. Thus the normals to these planes point along $[100]$, $[110]$, and $[111]$, respectively.

- (iii) The given intercepts are

$$x = a, \quad y = \frac{b}{2}, \quad z = 3c.$$

Measured in units of the corresponding axial lattice constants, the intercepts are

$$\left(1, \frac{1}{2}, 3\right).$$

Taking their reciprocals gives

$$\left(1, 2, \frac{1}{3}\right).$$

Multiplication by the least common multiple 3 clears the fraction:

$$3 \left(1, 2, \frac{1}{3}\right) = (3, 6, 1).$$

Since 3, 6, and 1 have no common integral factor, these are already the smallest relatively prime integers. Therefore the Miller indices are

$$\boxed{(361)}.$$

Equivalently, the plane equation obtained from the intercept form is

$$\frac{x}{a} + \frac{y}{b/2} + \frac{z}{3c} = 1,$$

or

$$3\frac{x}{a} + 6\frac{y}{b} + \frac{z}{c} = 3.$$

Its coefficients are proportional to $3 : 6 : 1$, confirming the same indices.

CONCLUSION:

A Bravais lattice is generated by integral combinations of primitive translations and has one translationally equivalent class of lattice points. A periodic non-Bravais arrangement requires a basis containing two or more inequivalent sites attached to each Bravais-lattice point.

In the stated positive-axis convention, the cubic planes are

$$(100) : x = a$$

$$(110) : x + y = a$$

$$(111) : x + y + z = a$$

with normals directed along $[100]$, $[110]$, and $[111]$, respectively. These relations assume a cubic cell with mutually perpendicular axes and equal lattice parameter a .

For intercepts a , $b/2$, and $3c$, the reciprocal-intercept construction gives the dimensionless, relatively prime Miller indices

$$(361).$$

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Q7. The spacing between successive (100) planes in sodium chloride is $1 \cdot 41 \text{ \AA}$. X-rays incident on the surface of the crystal are found to give rise to second order Bragg reflections at a glancing angle 10° . Calculate the wavelength of X-ray radiations.

INTRODUCTION:

The wavelength is determined using Bragg's law,

$$n\lambda = 2d \sin \theta,$$

where d is the spacing between successive reflecting planes, n is the order of reflection, and θ is the glancing angle between the incident X-ray beam and the crystal planes. Here,

$$d = 1.41 \text{ \AA}, \quad n = 2, \quad \theta = 10^\circ,$$

with $1 \text{ \AA} = 10^{-10} \text{ m}$.

SOLUTION:

Substituting the given values in Bragg's law,

$$2\lambda = 2(1.41 \text{ \AA}) \sin 10^\circ.$$

Hence,

$$\lambda = 1.41 \text{ \AA} \times \sin 10^\circ.$$

Using

$$\sin 10^\circ = 0.17365,$$

we obtain

$$\lambda = (1.41)(0.17365) \text{ \AA} = 0.24485 \text{ \AA}.$$

Therefore, to three significant figures,

$$\lambda = 0.245 \text{ \AA}.$$

In SI units,

$$\lambda = 0.245 \times 10^{-10} \text{ m} = 2.45 \times 10^{-11} \text{ m}.$$

The result also satisfies the Bragg-reflection condition $n\lambda \leq 2d$, since

$$2\lambda = 0.490 \text{ \AA} < 2d = 2.82 \text{ \AA}.$$

CONCLUSION:

The wavelength of the incident X-ray radiation is

$$\lambda = 0.245 \text{ \AA} = 2.45 \times 10^{-11} \text{ m}.$$

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Q8. Consider the planes with indices (1 0 0) and (0 0 1); the lattice is f.c.c., and the indices refer to the conventional cubic cell. What are the indices of these planes when referred to the primitive axes: $\vec{a}_1 = \frac{a}{2}(\hat{x} + \hat{y})$, $\vec{a}_2 = \frac{a}{2}(\hat{y} + \hat{z})$ and $\vec{a}_3 = \frac{a}{2}(\hat{z} + \hat{x})$?

INTRODUCTION:

The task is to transform the Miller indices from the conventional cubic basis

$$\vec{a} = a\hat{x}, \quad \vec{b} = a\hat{y}, \quad \vec{c} = a\hat{z}$$

to the given primitive f.c.c. basis. Let a position vector be written in primitive coordinates as

$$\vec{r} = u_1\vec{a}_1 + u_2\vec{a}_2 + u_3\vec{a}_3.$$

The Miller indices of a plane are obtained from the reciprocals of its intercepts on the chosen crystallographic axes, with the resulting triplet reduced to the smallest integers. Equivalently, if a plane has conventional indices (hkl) , its orientation is specified by the covector

$$\vec{G} = \frac{2\pi}{a}(h\hat{x} + k\hat{y} + l\hat{z}).$$

Its components with respect to the primitive reciprocal basis are determined by

$$h'_i = \frac{\vec{G} \cdot \vec{a}_i}{2\pi}.$$

SOLUTION:

For the given primitive vectors,

$$\vec{a}_1 = \frac{a}{2}(\hat{x} + \hat{y}), \quad \vec{a}_2 = \frac{a}{2}(\hat{y} + \hat{z}), \quad \vec{a}_3 = \frac{a}{2}(\hat{z} + \hat{x}),$$

the primitive reciprocal components of a conventional plane (hkl) are

$$\begin{aligned} h'_1 &= \frac{\vec{G} \cdot \vec{a}_1}{2\pi} = \frac{h+k}{2}, \\ h'_2 &= \frac{\vec{G} \cdot \vec{a}_2}{2\pi} = \frac{k+l}{2}, \\ h'_3 &= \frac{\vec{G} \cdot \vec{a}_3}{2\pi} = \frac{l+h}{2}. \end{aligned}$$

Thus,

$$(hkl)_{\text{cubic}} \longrightarrow \left(\frac{h+k}{2}, \frac{k+l}{2}, \frac{l+h}{2} \right)_{\text{primitive}}.$$

For the (100) plane,

$$(100)_{\text{cubic}} \longrightarrow \left(\frac{1}{2}, 0, \frac{1}{2} \right)_{\text{primitive}}.$$

Clearing the common denominator and reducing to the smallest integers gives

$$\left(\frac{1}{2}, 0, \frac{1}{2} \right) \sim (101).$$

The same result follows directly from the plane equation. Since

$$\vec{r} = \frac{a}{2} [(u_1 + u_3)\hat{x} + (u_1 + u_2)\hat{y} + (u_2 + u_3)\hat{z}],$$

the representative cubic plane $x = a$ becomes

$$\frac{a}{2}(u_1 + u_3) = a \implies u_1 + u_3 = 2.$$

Its primitive-axis intercepts are $(2, \infty, 2)$, whose reciprocals are

$$\left(\frac{1}{2}, 0, \frac{1}{2} \right) \sim (101).$$

For the (001) plane,

$$(001)_{\text{cubic}} \longrightarrow \left(0, \frac{1}{2}, \frac{1}{2} \right)_{\text{primitive}}.$$

Therefore,

$$\left(0, \frac{1}{2}, \frac{1}{2} \right) \sim (011).$$

Indeed, the representative plane $z = a$ gives

$$\frac{a}{2}(u_2 + u_3) = a \implies u_2 + u_3 = 2,$$

so its primitive-axis intercepts are $(\infty, 2, 2)$ and their reciprocals yield the same indices.

CONCLUSION:

When referred to the given primitive axes, the plane orientations are

$$(100)_{\text{cubic}} \equiv (101)_{\text{primitive}}$$

$$(001)_{\text{cubic}} \equiv (011)_{\text{primitive}}.$$

For the particular representative planes $x = a$ and $z = a$, the unreduced reciprocal-intercept triplets are respectively

$$\left(\frac{1}{2}, 0, \frac{1}{2} \right) \quad \text{and} \quad \left(0, \frac{1}{2}, \frac{1}{2} \right);$$

their conventional integer Miller-index forms are (101) and (011) .

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Q9. Derive diffraction conditions using reciprocal lattice concept. What are these conditions known as ?

INTRODUCTION:

Consider a crystal whose direct lattice is generated by the primitive translation vectors

$$\mathbf{a}_1, \quad \mathbf{a}_2, \quad \mathbf{a}_3,$$

so that every lattice point is at

$$\mathbf{R} = n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2 + n_3 \mathbf{a}_3, \quad n_1, n_2, n_3 \in \mathbb{Z}.$$

The reciprocal primitive vectors are defined, using the 2π convention, by

$$\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi \delta_{ij}.$$

Thus every reciprocal-lattice vector has the form

$$\mathbf{G} = h\mathbf{b}_1 + k\mathbf{b}_2 + l\mathbf{b}_3, \quad h, k, l \in \mathbb{Z}.$$

Let a monochromatic plane wave of wavelength λ be incident on the crystal with wave vector

$$\mathbf{k}_i = \frac{2\pi}{\lambda} \hat{\mathbf{s}}_i,$$

and let the elastically scattered wave have wave vector

$$\mathbf{k}_f = \frac{2\pi}{\lambda} \hat{\mathbf{s}}_f.$$

The scattering vector is taken to be

$$\mathbf{q} = \mathbf{k}_f - \mathbf{k}_i.$$

The required diffraction condition follows by demanding constructive interference between waves scattered from all translationally equivalent unit cells.

SOLUTION:

For an atom situated at $\mathbf{r}$, the incident wave contributes a phase factor

$$\exp(i\mathbf{k}_i \cdot \mathbf{r}),$$

whereas propagation of the scattered wave toward the observation direction contributes

$$\exp(-i\mathbf{k}_f \cdot \mathbf{r}).$$

Consequently, apart from the atomic scattering factor, the amplitude from that atom contains

$$\exp[-i(\mathbf{k}_f - \mathbf{k}_i) \cdot \mathbf{r}] = \exp(-i\mathbf{q} \cdot \mathbf{r}).$$

Let ρ_μ denote the position of the μ -th atom within a unit cell. The position of the corresponding atom in the cell translated by $\mathbf{R}$ is

$$\mathbf{r}_{\mu,\mathbf{R}} = \mathbf{R} + \rho_\mu.$$

The total scattered amplitude can therefore be written in the form

$$A(\mathbf{q}) = \sum_{\mathbf{R}} \sum_{\mu} f_{\mu}(\mathbf{q}) \exp[-i\mathbf{q} \cdot (\mathbf{R} + \rho_{\mu})].$$

Separating the unit-cell and lattice contributions gives

$$A(\mathbf{q}) = \sum_{\mu} f_{\mu}(\mathbf{q}) e^{-i\mathbf{q} \cdot \rho_{\mu}} \sum_{\mathbf{R}} e^{-i\mathbf{q} \cdot \mathbf{R}},$$

$\underbrace{\hspace{1.5cm}}_{F(\mathbf{q})}$

$\underbrace{\hspace{1.5cm}}_{S(\mathbf{q})}$

where $F(\mathbf{q})$ is the structure factor and $S(\mathbf{q})$ is the lattice sum.

For a crystal containing N_1, N_2, N_3 cells along the three primitive directions,

$$S(\mathbf{q}) = \prod_{j=1}^3 \left[\sum_{n_j=0}^{N_j-1} e^{-in_j \mathbf{q} \cdot \mathbf{a}_j} \right].$$

Each factor is a geometric series:

$$\sum_{n=0}^{N-1} e^{-inx} = e^{-i(N-1)x/2} \frac{\sin(Nx/2)}{\sin(x/2)}.$$

Its magnitude becomes of order N , rather than remaining bounded, when

$$x = 2\pi m, \quad m \in \mathbb{Z}.$$

Hence sharp diffraction maxima occur when

$$\mathbf{q} \cdot \mathbf{a}_1 = 2\pi h, \quad \mathbf{q} \cdot \mathbf{a}_2 = 2\pi k, \quad \mathbf{q} \cdot \mathbf{a}_3 = 2\pi l,$$

where h, k, l are integers.

Expanding $\mathbf{q}$ in the reciprocal basis,

$$\mathbf{q} = x_1 \mathbf{b}_1 + x_2 \mathbf{b}_2 + x_3 \mathbf{b}_3,$$

and taking scalar products with the direct primitive vectors gives

$$\mathbf{a}_1 \cdot \mathbf{q} = 2\pi x_1, \quad \mathbf{a}_2 \cdot \mathbf{q} = 2\pi x_2, \quad \mathbf{a}_3 \cdot \mathbf{q} = 2\pi x_3.$$

Comparison with the constructive-interference conditions yields

$$x_1 = h, \quad x_2 = k, \quad x_3 = l.$$

Therefore,

$$\mathbf{k}_f - \mathbf{k}_i = \mathbf{q} = \mathbf{G} = h\mathbf{b}_1 + k\mathbf{b}_2 + l\mathbf{b}_3.$$

Thus diffraction occurs only when the momentum-transfer wave vector is a reciprocal-lattice vector.

Equivalently, writing

$$\mathbf{k}_i = \frac{2\pi}{\lambda} \hat{\mathbf{s}}_i, \quad \mathbf{k}_f = \frac{2\pi}{\lambda} \hat{\mathbf{s}}_f,$$

the three scalar conditions become

$$\mathbf{a}_1 \cdot (\hat{\mathbf{s}}_f - \hat{\mathbf{s}}_i) = h\lambda,$$

$$\mathbf{a}_2 \cdot (\hat{\mathbf{s}}_f - \hat{\mathbf{s}}_i) = k\lambda,$$

$$\mathbf{a}_3 \cdot (\hat{\mathbf{s}}_f - \hat{\mathbf{s}}_i) = l\lambda.$$

These are the three scalar forms of the diffraction condition.

For elastic diffraction,

$$|\mathbf{k}_f| = |\mathbf{k}_i| = \frac{2\pi}{\lambda}.$$

Since $\mathbf{k}_f = \mathbf{k}_i + \mathbf{G}$, this requires

$$|\mathbf{k}_i + \mathbf{G}|^2 = |\mathbf{k}_i|^2,$$

and hence

$$2\mathbf{k}_i \cdot \mathbf{G} + G^2 = 0.$$

Geometrically, the endpoint of $\mathbf{k}_f$ must lie both on a sphere of radius $2\pi/\lambda$, centred at the tail of $\mathbf{k}_i$, and at a reciprocal-lattice point. This is the reciprocal-space construction of the diffraction condition, commonly called the Ewald-sphere construction.

The equivalence with Bragg's law follows by associating $\mathbf{G}$ with a family of lattice planes normal to $\mathbf{G}$. If their spacing is d , then

$$|\mathbf{G}| = \frac{2\pi}{d}.$$

Let θ be the glancing angle between the incident beam and these planes. For the reflecting orientation,

$$\mathbf{k}_i \cdot \mathbf{G} = -|\mathbf{k}_i||\mathbf{G}| \sin \theta.$$

Substitution in

$$2\mathbf{k}_i \cdot \mathbf{G} + G^2 = 0$$

gives

$$2|\mathbf{k}_i||\mathbf{G}| \sin \theta = G^2.$$

Using

$$|\mathbf{k}_i| = \frac{2\pi}{\lambda}, \quad |\mathbf{G}| = \frac{2\pi}{d},$$

one obtains

$$2d \sin \theta = \lambda.$$

If the momentum transfer is the n -th multiple of the reciprocal vector corresponding to planes of spacing d , the result is

$$2d \sin \theta = n\lambda,$$

which is Bragg's law.

The lattice condition determines the allowed positions of diffraction maxima, while the structure factor $F(\mathbf{G})$ determines their relative intensities and may make some otherwise allowed reflections vanish.

CONCLUSION:

The reciprocal-lattice diffraction condition is

$$\mathbf{k}_f - \mathbf{k}_i = \mathbf{G}, \quad \mathbf{G} = h\mathbf{b}_1 + k\mathbf{b}_2 + l\mathbf{b}_3, \quad h, k, l \in \mathbb{Z},$$

or, equivalently,

$$\mathbf{a}_1 \cdot (\mathbf{k}_f - \mathbf{k}_i) = 2\pi h, \quad \mathbf{a}_2 \cdot (\mathbf{k}_f - \mathbf{k}_i) = 2\pi k, \quad \mathbf{a}_3 \cdot (\mathbf{k}_f - \mathbf{k}_i) = 2\pi l.$$

These are known as the *Laue diffraction conditions* or *Laue equations*. For elastic scattering they are equivalent to Bragg's law,

$$2d \sin \theta = n\lambda.$$

Physically, a diffraction maximum is possible when the scattering vector terminates at a reciprocal-lattice point; its observable intensity is then governed by the unit-cell structure factor.

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Q10. Draw the crystal structure of CaF_2 indicating the positions of cations and anions. What are their respective coordination numbers?

INTRODUCTION:

CaF_2 crystallizes in the cubic fluorite structure. We describe one conventional cubic unit cell of edge length a , with Cartesian axes along its edges and the origin at a corner. Fractional coordinates (x, y, z) denote the position

$$\mathbf{r} = a(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}).$$

The Ca^{2+} ions form a face-centred cubic lattice, while the F^- ions occupy all the tetrahedral interstitial sites of that lattice. The required coordination numbers follow by identifying the nearest neighbours of each ion.

SOLUTION:

The Ca^{2+} ions occupy the corners and face centres of the conventional cubic cell. Representative fractional coordinates are

$$(0, 0, 0), \quad \left(0, \frac{1}{2}, \frac{1}{2}\right), \quad \left(\frac{1}{2}, 0, \frac{1}{2}\right), \quad \left(\frac{1}{2}, \frac{1}{2}, 0\right),$$

together with lattice-equivalent positions on the cell boundary.

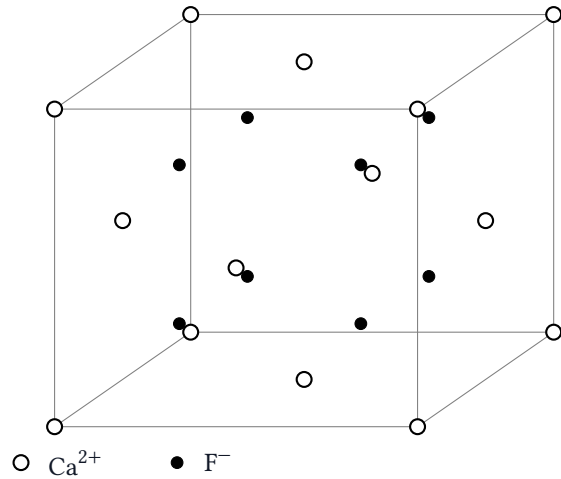
All eight tetrahedral sites are occupied by F^- ions:

$$(x, y, z), \quad x, y, z \in \left\{\frac{1}{4}, \frac{3}{4}\right\}.$$

Thus the fluorine-ion positions include

$$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right), \left(\frac{1}{4}, \frac{3}{4}, \frac{1}{4}\right), \left(\frac{3}{4}, \frac{1}{4}, \frac{1}{4}\right),$$

and the four positions obtained by the remaining choices of $\frac{1}{4}$ and $\frac{3}{4}$.



The number of calcium ions belonging to one conventional cell is

$$8\left(\frac{1}{8}\right) + 6\left(\frac{1}{2}\right) = 4,$$

whereas all eight fluoride ions shown lie inside the cell:

$$N_{\text{F}} = 8.$$

Hence

$$N_{\text{Ca}} : N_{\text{F}} = 4 : 8 = 1 : 2,$$

as required by the formula CaF_2 .

For a calcium ion at the origin, the nearest fluoride ions are at relative positions

$$\frac{a}{4}(\pm 1, \pm 1, \pm 1).$$

There are $2^3 = 8$ such choices, all at the same distance

$$d_{\text{Ca-F}} = \sqrt{\left(\frac{a}{4}\right)^2 + \left(\frac{a}{4}\right)^2 + \left(\frac{a}{4}\right)^2} = \frac{\sqrt{3}a}{4}.$$

Therefore each Ca^{2+} ion is surrounded by eight F^- ions situated at the vertices of a cube.

Consider next the fluoride ion at

$$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

Its four nearest calcium ions are at

$$(0, 0, 0), \quad \left(0, \frac{1}{2}, \frac{1}{2}\right), \quad \left(\frac{1}{2}, 0, \frac{1}{2}\right), \quad \left(\frac{1}{2}, \frac{1}{2}, 0\right).$$

Each is at distance $\sqrt{3}a/4$ from the fluoride ion, and these four calcium ions form the vertices of a tetrahedron. Thus each F^- ion has four nearest Ca^{2+} neighbours.

The coordination counts are also mutually consistent: with four calcium and eight fluoride ions per cell,

$$4(8) = 8(4) = 32,$$

so both descriptions count the same number of nearest-neighbour Ca – F contacts.

CONCLUSION:

In the fluorite structure of CaF_2 , the Ca^{2+} ions form a face-centred cubic lattice and the F^- ions occupy all eight tetrahedral sites of each conventional unit cell. Each Ca^{2+} ion is cubically coordinated by eight F^- ions, whereas each F^- ion is tetrahedrally coordinated by four Ca^{2+} ions. Therefore,

$CN(Ca^{2+}) = 8, \quad CN(F^-) = 4.$



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